Depth of Focus

overall phase term is

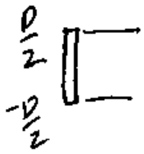
$$\underbrace{e^{-j\frac{k}{2z_0}(x^2+y^2)}}_{\text{lens}} \; \underbrace{e^{+j\frac{k}{2z}(x^2+y^2)}}_{\text{propagation}}$$

for $z \neq z_0$ lens does not perfectly cancel out the phase due to propagation

consider x-direction only

$$\Delta\varphi = \frac{kx^2}{2}\left(\frac{1}{z_0} - \frac{1}{z}\right)$$

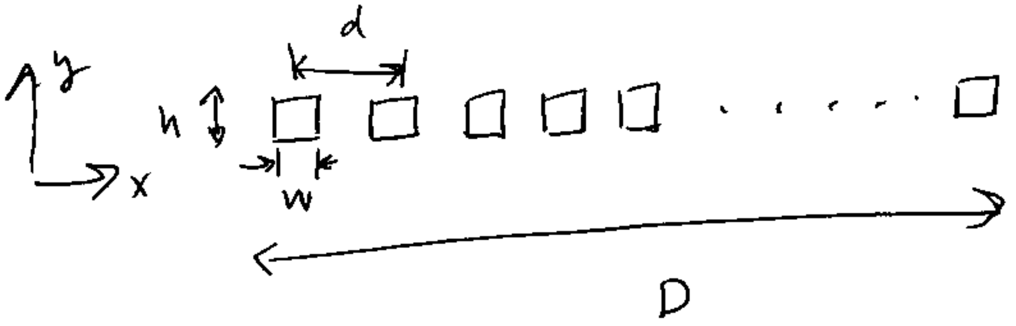
over geometric extent of transducer $\frac{x^2}{2} = \frac{D^2}{4}$



$$\Delta\phi = \left|k\frac{D^2}{4}\left(\frac{1}{z_0} - \frac{1}{z}\right)\right| < 1 \quad \text{radian.}$$

$$\left|\frac{1}{z_0} - \frac{1}{z}\right| < \frac{4\lambda}{D^2 2\pi} = \frac{2\lambda}{\pi D^2}$$

Phased-Array



$$s(x,y) = \mathrm{rect}\left(\frac{y}{h}\right)\left[\mathrm{rect}\left(\frac{x}{D}\right)\frac{1}{d}\,\mathrm{comb}\left(\frac{x}{d}\right)\right] * \mathrm{rect}\left(\frac{x}{w}\right)$$

Far Field Response

(ignoring leading phase factors)

$$h(x_0,y_0) = \mathcal{F}\left(s(x,y)\right)\Big|_{\frac{x_0}{\lambda z},\,\frac{y_0}{\lambda z}}$$

$$= \left[\mathrm{sinc}(w k_x)\cdot w\right]\cdot\left[D\,\mathrm{sinc}(D k_x) * \mathrm{comb}(k_x d)\right] h\,\mathrm{sinc}(k_y h)$$

$$= wDh\underbrace{\left[\mathrm{sinc}\left(\frac{D x_0}{\lambda z}\right) * \mathrm{comb}\left(\frac{d x_0}{\lambda z}\right)\right]}\,\mathrm{sinc}\left(w\frac{x_0}{\lambda z}\right)\mathrm{sinc}\left(h\frac{y_0}{\lambda z}\right)$$

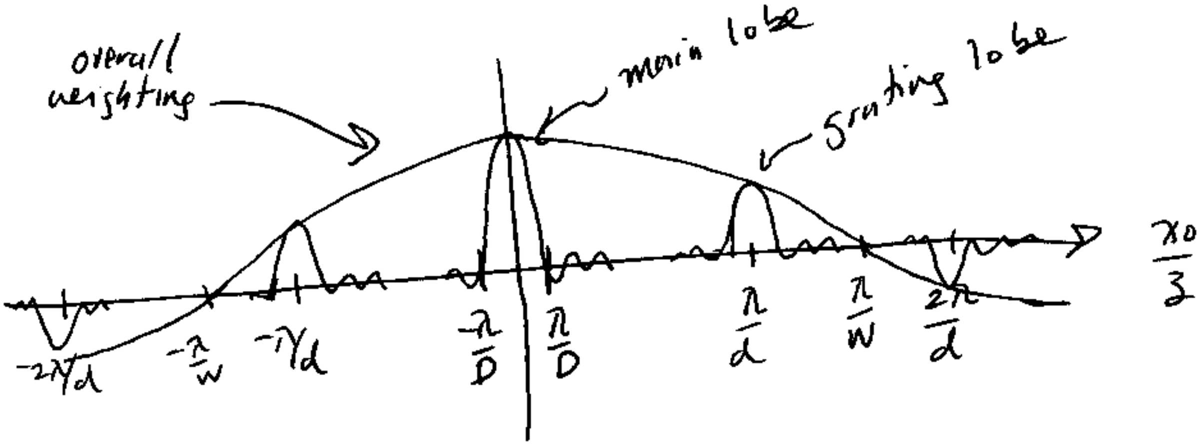
$$\mathrm{sinc}\left(\frac{D x_0}{\lambda z}\right) * \sum_{n=-\infty}^{\infty}\delta\left(\frac{d x_0}{\lambda z} - n\right)$$

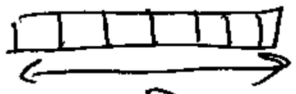
$$= \mathrm{sinc}\left(\frac{D x_0}{\lambda z}\right) * \frac{1}{d}\sum \delta\left(\frac{x_0}{\lambda z} - \frac{n}{d}\right)$$

$$= \frac{1}{d}\sum \mathrm{sinc}\left(D\left(\frac{x_0}{\lambda z} - \frac{n}{d}\right)\right)$$

Phased Array

$$h(x_0, y_0) = \frac{WDh}{d} \operatorname{sinc}\left(W \frac{x_0}{\lambda z}\right) \sum \operatorname{sinc}\left(D\left(\frac{x_0}{\lambda z} - \frac{n}{d}\right)\right) \operatorname{sinc}\left(\frac{l y_0}{\lambda z}\right)$$



- If $d = W$, then the array is continuous,  and we just have a main lobe $\propto \operatorname{sinc}\left(\frac{D x_0}{\lambda z}\right)$

- In the far field $\frac{x_0}{z} \approx \sin\theta \approx \theta$ (angle of beam)

so main lobe width $\frac{D}{\lambda}\sin\theta = 1$

$$\sin\theta = \frac{\lambda}{D}$$

$$\theta = \operatorname{asin}\left(\frac{\lambda}{D}\right)$$

Example: $w = d = \lambda/2$
128 elements
$D = 64\lambda$
$\theta = \operatorname{asin}\left(\frac{1}{64}\right) = 0.89$ degrees

FOCUSING

Now consider the beam pattern from the phased array in the near field (Fresnel zone)

Recall expression is

$$U = \frac{\exp(jv)}{j\lambda z} \mathcal{F}\left[s(x,y)\, e^{\frac{jk}{2z}(x^2+y^2)} \right]$$

$$s(x,y) = \mathrm{rect}\left(\frac{y}{h}\right)\left[\mathrm{rect}\left(\frac{x}{D}\right) \frac{1}{d}\,\mathrm{comb}\left(\frac{x}{d}\right)\right] * \mathrm{rect}\left(\frac{x}{w}\right)$$

For now, ignore y-dependence and concentrate on x-dependence.

We can write the array pattern as

$$s(x) = \sum_{n=-\frac{N}{2}}^{N/2} \mathrm{rect}\left(\frac{x-nd}{w}\right)$$

Then ~~the~~

$$s(x)\, e^{\frac{jk}{2z}x^2} = \left[\sum_{n=-\frac{N}{2}}^{N/2} \mathrm{rect}\left(\frac{x-nd}{w}\right)\right] \exp\left(\frac{jk}{2z}x^2\right)$$

Now let us consider adding a phase term to the n^{th} element of the array.

To cancel the $\exp\left(\frac{jk}{2z}x^2\right)$ term, this term should be equal to $\exp\left(-\frac{jk}{2z}(nd)^2\right)$

FOCUSING

How do we achive the desired phase profile?

Recall, our pulses are of the form:

$$p(t) = \underbrace{a(t)}_{\text{envelope}} \underbrace{\cos(\omega_0 t)}_{RF}$$

Delayed pulse is $p(t-\frac{r}{c}) = p(t-\frac{r}{c})\cos(\omega_0(t-\frac{r}{c}))$

To simplify the analysis, we use complex notation

$$p(t) = \text{Real}(\tilde{p}(t)) \text{ where } \tilde{p}(t) = a(t)e^{-j\omega_0 t}$$

$$\tilde{p}(t) = a(t-\frac{r}{c})e^{-j\omega_0(t-r/c)}$$

$$= a(t-\frac{r}{c})e^{-j\frac{2\pi f_0}{c}(ct-r)}$$

$$= a(t-\frac{r}{c})e^{-jkct}e^{+jkr}$$

Now, consider delaying the pulse

$$\tilde{p}(t-\tau) = a(t-\tau-\frac{r}{c})e^{-jkct}e^{+jkc\tau}e^{jkr}$$

So ~~ne~~ phase induced by delay is $\underline{\underline{kc\tau}}$

We want $kc\tau = -\frac{k}{2z}(nd)^2$

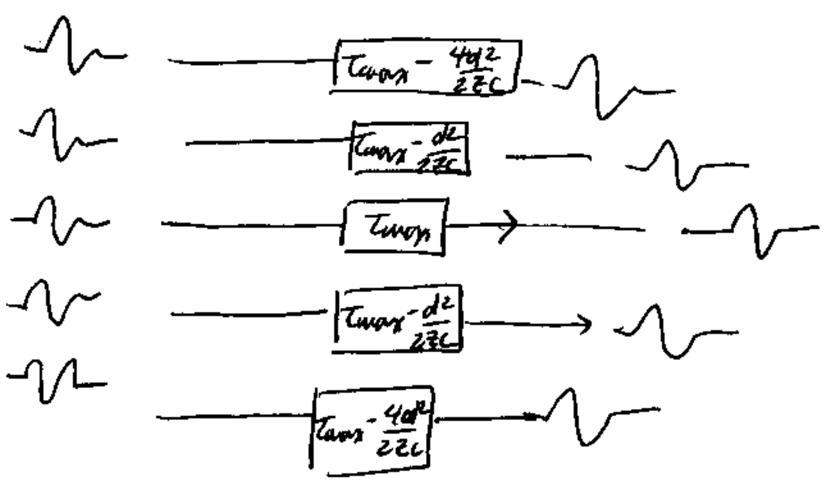
$$\tau = -\frac{(nd)^2}{2zc}$$

FOCUSING

$$\tau = -\frac{(nd)^2}{2zc}$$ ⇒ implies we need negative delays.

In practice, delays $> \phi$. So we define

$$\tau = \tau_{max} - \frac{(nd)^2}{2zc} > \phi$$



In practice, focusing achieved by a coarse time delay and a fine time delay / phase shift.

Beam Steering

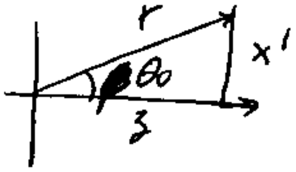
Recall, we can steer the beam by putting a phase term of the form $\exp(j2\pi \frac{x'}{\lambda_0 z} x)$ across the transducer.

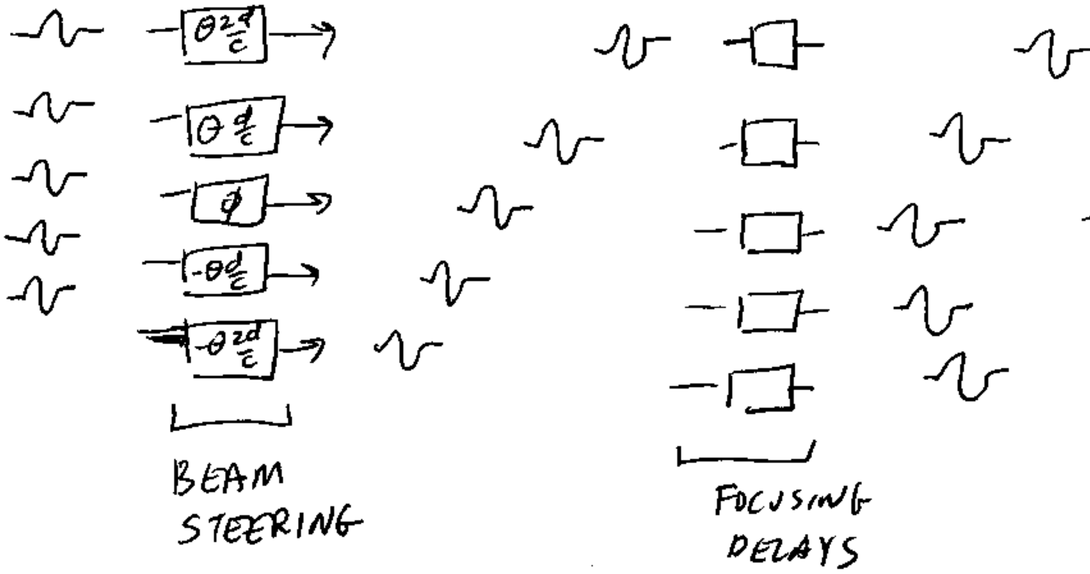
$$= \exp(j k \frac{x'}{z} x)$$

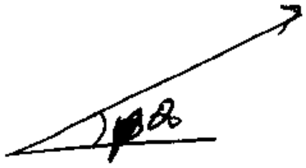
So, we want

$$kc\tau = \frac{k x' x}{z}$$

$$\tau = \frac{x' nd}{zc} \approx \beta \frac{nd}{c} \qquad \beta = \frac{x'}{z} \approx \theta_0$$







Total delays

$$\tau = \tau_{max} - \frac{(nd)^2}{2zc} + \frac{\beta nd}{c}$$