Depth of Focus

Overall phase term is
\[
\frac{e^{-j\frac{k}{2z_0} (x^2 + y^2)}}{\text{lens}} \cdot \frac{e^{j\frac{k}{2z} (x^2 + y^2)}}{\text{propagation}}
\]

for \( z \neq z_0 \) lens does not perfectly cancel out the phase due to propagation.

Consider x-direction only
\[
\Delta \phi = \frac{k x^2}{z} \left( \frac{1}{z_0} - \frac{1}{z} \right)
\]

over geometric extent of transducer \( \frac{x^2}{2} = \frac{D^2}{4} \)

\[
\Delta \phi = \left| \frac{k D^2}{4} \left( \frac{1}{z_0} - \frac{1}{z} \right) \right| < 1 \text{ radian}
\]

\[
\left| \frac{1}{z_0} - \frac{1}{z} \right| < \frac{4 \lambda}{D^2 2 \pi} = \frac{2 \lambda}{\pi D^2}
\]
Phased-Array

\[ s(x, y) = \text{rect}(\frac{y}{h}) \left[ \text{rect}(\frac{x}{D}) \frac{1}{d} \text{comb}(\frac{x}{d}) \right] \ast \text{rect}(\frac{x}{W}) \]

**Far Field Response**

(Ignoing leading phase factors)

\[ h(x_0, y_0) = \mathcal{F}(s(x, y)) \bigg|_{x_0, y_0}^{\frac{x_0}{\lambda z}, \frac{y_0}{\lambda z}} \]

\[ = \left[ \text{sinc}(Wk_x) \cdot W \right] \ast \left[ D \text{sinc}(Dk_x) \ast \text{comb}(k_xd) \right] h \text{sinc}(k_yh) \]

\[ = WDh \left[ \text{sinc}(D\frac{x_0}{\lambda z}) \ast \text{comb}(\frac{dx_0}{\lambda z}) \right] \text{sinc}(W \frac{x_0}{\lambda z}) \text{sinc}(h \frac{y_0}{\lambda z}) \]

\[ \text{sinc} \left( D \frac{x_0}{\lambda z} \right) \ast \sum_{n=-\infty}^{\infty} \delta \left( \frac{dx_0}{\lambda z} - n \right) \]

\[ = \text{sinc} \left( D \frac{x_0}{\lambda z} \right) \frac{1}{d} \sum_{n=-\infty}^{\infty} \delta \left( \frac{x_0}{\lambda z} - \frac{n}{d} \right) \]

\[ = \frac{1}{d} \sum_{n=-\infty}^{\infty} \text{sinc} \left( D \left( \frac{x_0}{\lambda z} - \frac{n}{d} \right) \right) \]
**Phased Array**

\[ h(x_0, y_0) = \frac{WDh}{\lambda} \sin c \left( \frac{Wx_0}{\lambda} \right) \sum \sin c \left( \frac{D}{\lambda} \left( \frac{y_0}{\lambda} - \frac{\pi}{2} \right) \right) \sin c \left( \frac{2y_0}{\lambda} \right) \]

- If \( d = w \), then the array is continuous, and we just have a main lobe \( \propto \sin c \left( \frac{Dx_0}{\lambda} \right) \).

- In the far field \( \frac{x_0}{\lambda} \approx \sin \theta \propto \theta \) (angle of beam)

  \[ \text{so main lobe width} \quad \frac{D}{\lambda} \sin \theta = 1 \]

  \[ \sin \theta = \frac{\lambda}{D} \]

  \[ \theta = \arcsin \left( \frac{\lambda}{D} \right) \]

**Example:**
- \( w = d = \frac{\lambda}{2} \)
- 128 elements
- \( D = 64 \lambda \)
- \( \theta = \arcsin \left( \frac{1}{64} \right) = 0.89 \text{ degrees} \)
Now consider the beam pattern from the phased array in the near-field (Fraunhofer zone).

Recall the expression is

\[ U = \exp\left( \frac{i\pi}{2} \right) \int S(x, y) e^{\frac{ik}{2z}(x^2 + y^2)} \, dx \, dy \]

\[ S(x, y) = \text{rect}\left( \frac{y}{d} \right) \left[ \text{rect}\left( \frac{x}{d/2} \right) \frac{1}{d} \cos\left( \frac{2\pi}{d} \right) \right] \text{rect}\left( \frac{x}{w} \right) \]

for now, ignore \( y \)-dependence, and concentrate on \( x \)-dependence.

We can write the array pattern as

\[ S(x) = \sum_{n = -N/2}^{N/2} \text{rect}\left( \frac{x - nd}{w} \right) \]

Then

\[ S(x) e^{\frac{ik}{2z}x^2} = \left[ \sum_{n = -N/2}^{N/2} \text{rect}\left( \frac{x - nd}{w} \right) \right] \exp\left( \frac{i\pi}{2z} x^2 \right) \]

Now let us consider adding a phase term to the \( n \)-th element of the array.

To cancel the \( \exp\left( \frac{i\pi}{2z} x^2 \right) \) term, this term should be equal to \( \exp\left( -\frac{i\pi}{2z} (nd)^2 \right) \).
Focusing

How do we achieve the desired phase profile?

Recall, our pulses are of the form:

$$p(t) = \frac{a(t) \cos(\omega_0 t)}{\text{envelope}_RF}$$

Delayed pulse is:

$$p(t-\frac{\tau}{c}) = p(t-\frac{\tau}{c}) \cos(\omega_0 (t-\frac{\tau}{c}))$$

To simplify the analysis, we use complex notation:

$$p(t) = \text{Real}[\tilde{p}(t)] \text{ where } \tilde{p}(t) = a(t) e^{-j\omega_0 t}$$

$$\tilde{p}(t) = a(t-\frac{\tau}{c}) e^{-j\omega_0 (t-\frac{\tau}{c})}$$

$$= a(t-\frac{\tau}{c}) e^{-j\frac{2\pi f_0}{c} (ct-\tau)}$$

$$= a(t-\frac{\tau}{c}) e^{-j kct} e^{jkr}$$

Now, consider delaying the pulse:

$$\tilde{p}(t-\tau) = a(t-\tau-\frac{\tau}{c}) e^{-j kct} e^{j kr} e^{jkr}$$

So, the phase induced by delay is $$\frac{kct}{c}$$

We want $$kct = \frac{k}{2\epsilon} (nd)^2$$

So:

$$\tau = -\frac{(nd)^2}{2\epsilon c}$$
Focusing

\[ T = -\frac{(nd)^2}{2zc} \] => implict we need negative delays.

In practice, delays \( > 0 \). So we define

\[ T = T_{\text{max}} - \frac{(nd)^2}{2zc} \] \( > 0 \)

In practice, focusing achieved by a coarse time delay and a fine time delay/phase shift.
Beam Steering

Recall, we can steer the beam by putting a phase term of the form $\exp(j 2\pi \frac{x'}{\lambda_0^2} x)$ across the transducers.

So, we want:

$$k_c z = \frac{k x' x}{2}$$

$$z = \frac{x' nd}{2c} + \frac{\beta nd}{c}$$

$$\beta = \frac{x'}{\lambda_0^2} \approx \theta_0$$

**Total delays**

$$z = T_{\text{max}} - \frac{(nd)^2}{2zc} + \frac{\beta nd}{c}$$