Bioengineering 280A
Principles of Biomedical Imaging

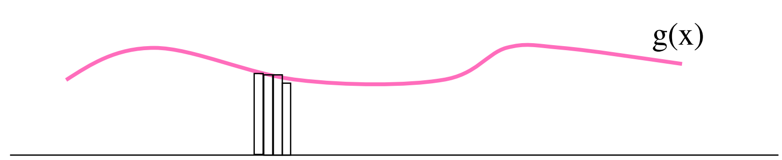
Fall Quarter 2005
Linear Systems Lecture 2

Representation of 1D Function

From the sifting property, we can write a 1D function as

$g(x) = \int_{-\infty}^{\infty} g(\xi)\delta(x-\xi)d\xi$. To gain intuition, consider the approximation

$g(x) \approx \sum_{n=-\infty}^{\infty} g(n\Delta x)\frac{1}{\Delta x}\Pi\left(\frac{x-n\Delta x}{\Delta x}\right)\Delta x.$



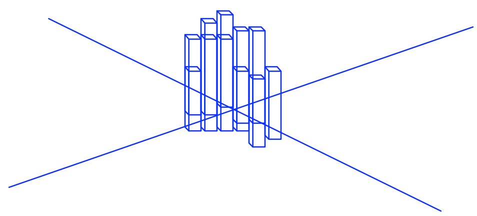
Representation of 2D Function

Similarly, we can write a 2D function as

$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x-\xi,y-\eta)d\xi d\eta.$$

To gain intuition, consider the approximation

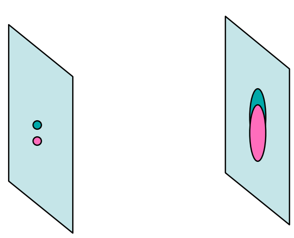
$$g(x,y) \approx \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} g(n\Delta x, m\Delta y)\frac{1}{\Delta x}\Pi\left(\frac{x-n\Delta x}{\Delta x}\right)\frac{1}{\Delta y}\Pi\left(\frac{y-m\Delta y}{\Delta y}\right)\Delta x\Delta y.$$



Impulse Response

Intuition: the impulse response is the response of a system to an input of infinitesimal width and unit area.

Original Image



Blurred Image

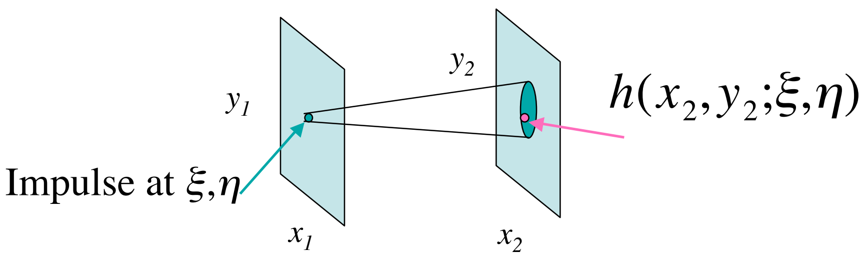
Since any input can be thought of as the weighted sum of impulses, a linear system is characterized by its impulse response(s).

Impulse Response

The impulse response characterizes the response of a system over all space to a Dirac delta impulse function at a certain location.

$h(x_2;\xi) = L[\delta(x_1 - \xi)]$ 1D Impulse Response

$h(x_2,y_2;\xi,\eta) = L[\delta(x_1 - \xi, y_1 - \eta)]$ 2D Impulse Response



Superposition Integral

What is the response to an arbitrary function $g(x_1,y_1)$?

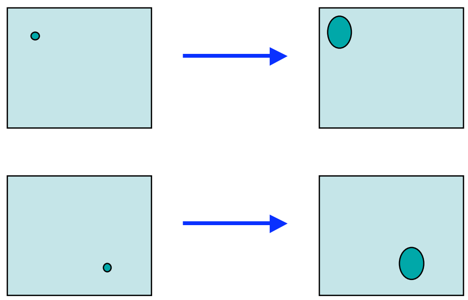
Write $g(x_1,y_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x_1 - \xi, y_1 - \eta)d\xi d\eta.$

The response is given by

$$\begin{aligned} I(x_2,y_2) &= L[g_1(x_1,y_1)] \\ &= L\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x_1 - \xi, y_1 - \eta)d\xi d\eta\right] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)L[\delta(x_1 - \xi, y_1 - \eta)]d\xi d\eta \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2,y_2;\xi,\eta)d\xi d\eta \end{aligned}$$

Space Invariance

If a system is space invariant, the impulse response depends only on the difference between the output coordinates and the position of the impulse and is given by $h(x_2,y_2;\xi,\eta) = h(x_2-\xi, y_2-\eta)$



2D Convolution

For a space invariant linear system, the superposition integral becomes a convolution integral.

$$
\begin{aligned}
I(x_2,y_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2,y_2;\xi,\eta)d\xi d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2-\xi,y_2-\eta)d\xi d\eta \\
&= g(x_2,y_2)**h(x_2,y_2)
\end{aligned}
$$

where ** denotes 2D convolution. This will sometimes be abbreviated as *, e.g. $I(x_2, y_2)= g(x_2, y_2)*h(x_2, y_2)$.

1D Convolution

For completeness, here is the 1D version.

$$\begin{aligned} I(x) &= \int_{-\infty}^{\infty} g(\xi)h(x;\xi)d\xi \\ &= \int_{-\infty}^{\infty} g(\xi)h(x-\xi)d\xi \\ &= g(x)*h(x) \end{aligned}$$

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Convolution with Dirac delta function

$$\begin{aligned} g(x)*\delta(x) &= \int_{-\infty}^{\infty} g(\xi)\delta(x-\xi)d\xi \\ &= g(x) \end{aligned}$$

$$\begin{aligned} g(x)*\delta(x-\Delta) &= \int_{-\infty}^{\infty} g(\xi)\delta(x-\Delta-\xi)d\xi \\ &= g(x-\Delta) \end{aligned}$$

Convolution of g(x) with a shifted Dirac delta function just shifts g(x)

$$\begin{aligned} g(x,y)**\delta(x,y) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x-\xi,y-\eta)d\xi d\eta \\ &= g(x,y) \end{aligned}$$

$$\begin{aligned} g(x,y)**\delta(x-x_0,y-y_0) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x-x_0-\xi,y-y_0-\eta)d\xi d\eta \\ &= g(x-x_0,y-y_0) \end{aligned}$$

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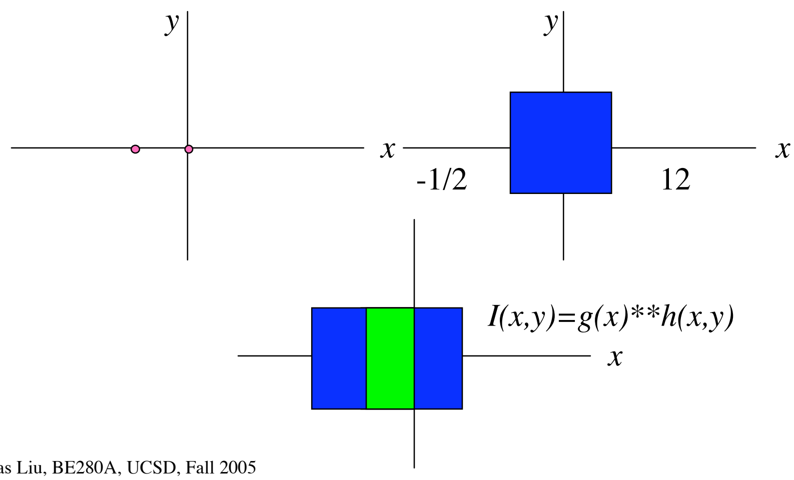
2D Convolution Example

$g(x) = \delta(x+1/2, y) + \delta(x,y)$ $h(x) = rect(x,y)$



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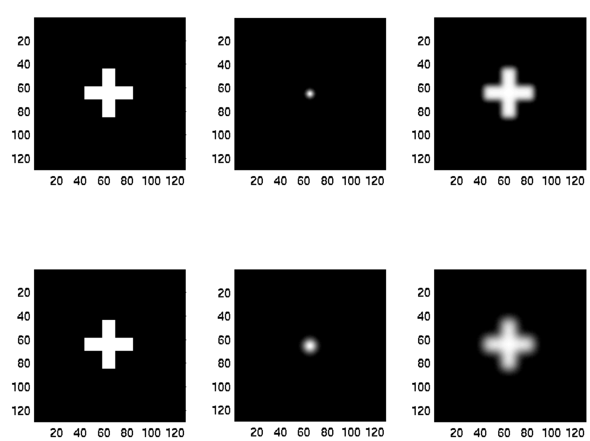
2D Convolution Example



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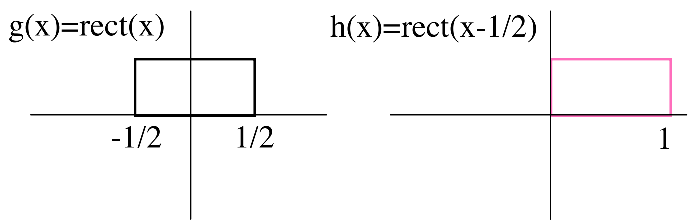
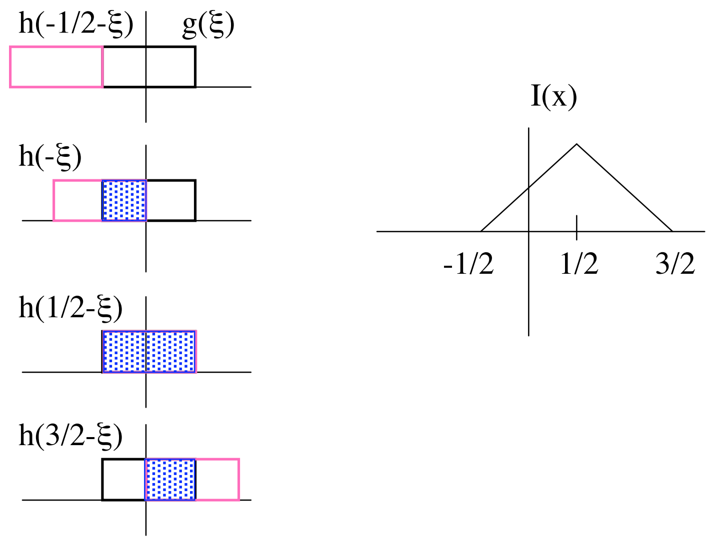
1D Convolution Review

$$g(x) * h(x) = \int_{-\infty}^{\infty} g(\xi)h(x-\xi)d\xi$$

Basic Rule: Flip one function, slide it past the other function, and integrate as you go.



1D Convolution Review



Convolution/Modulation Theorem

$$F\{g(x)*h(x)\} = \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} g(u)*h(x-u)du\right]e^{-j2\pi k_x x}dx$$
$$= \int_{-\infty}^{\infty} g(u)\int_{-\infty}^{\infty} h(x-u)e^{-j2\pi k_x x}dxdu$$
$$= \int_{-\infty}^{\infty} g(u)H(k_x)e^{-j2\pi k_x u}du$$
$$= G(k_x)H(k_x)$$

Convolution in the spatial domain transforms into multiplication in the frequency domain. Dual is modulation

$$F\{g(x)h(x)\} = G(k_x)*H(k_x)$$

2D Convolution/Multiplication

Convolution

$$F[g(x,y)**h(x,y)] = G(k_x,k_y)H(k_x,k_y)$$

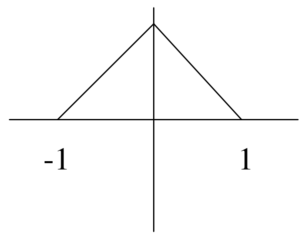
Multiplication

$$F[g(x,y)h(x,y)] = G(k_x,k_y)**H(k_x,k_y)$$

Application of Convolution Thm.

$$\Lambda(x) = \begin{cases} 1-|x| & |x|<1 \\ 0 & otherwise \end{cases}$$

$$F(\Lambda(x)) = \int_{-1}^{1}(1-|x|)e^{-j2\pi k_x x}dx = ??$$



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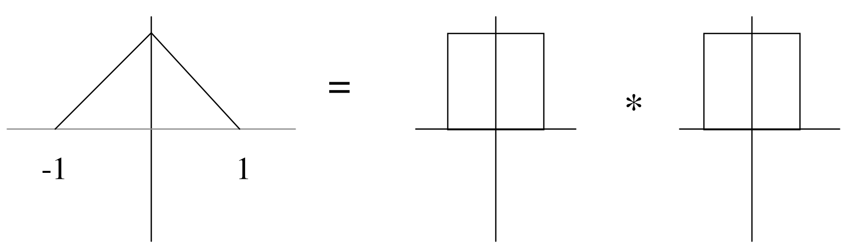
Application of Convolution Thm.

$$\Lambda(x) = \Pi(x) * \Pi(x)$$
$$F(\Lambda(x)) = \sin c^2(k_x)$$



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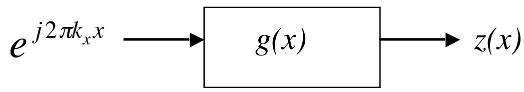
Eigenfunctions

The fundamental nature of the convolution theorem may be better understood by observing that the complex exponentials are eigenfunctions of the convolution operator.

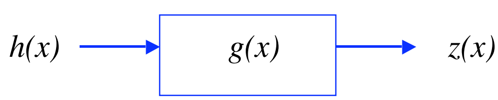


$$
\begin{aligned}
z(x) &= g(x) * e^{j2\pi k_x x} \\
&= \int_{-\infty}^{\infty} g(u) e^{j2\pi k_x (x-u)} du \\
&= G(k_x) e^{j2\pi k_x x}
\end{aligned}
$$

The response of a linear shift invariant system to a complex exponential is simply the exponential multiplied by the FT of the system's impulse response.

Eigenfunctions

Now consider an arbitrary input *h(x)*.



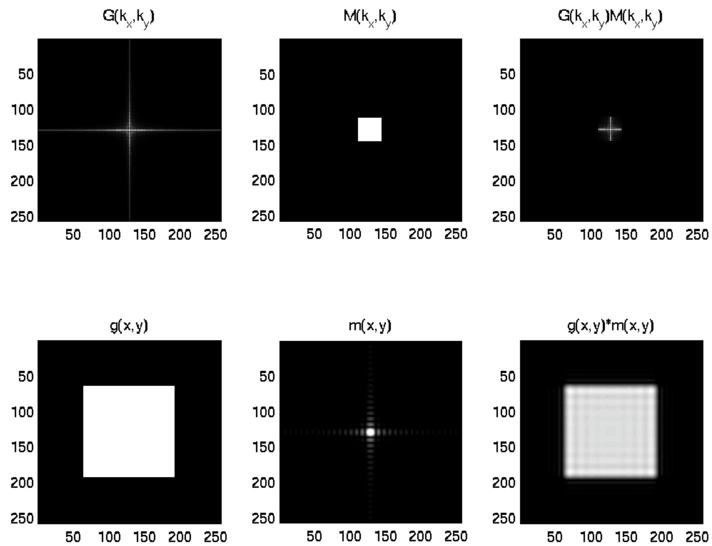
Recall that we can express *h(x)* as the integral of weighted complex exponentials.

$$h(x) = \int_{-\infty}^{\infty} H(k_x) e^{j2\pi k_x x} dk_x$$

Each of these exponentials is weighted by $G(k_x)$ so that the response may be written as

$$z(x) = \int_{-\infty}^{\infty} G(k_x) H(k_x) e^{j2\pi k_x x} dk_x$$

Convolution Example

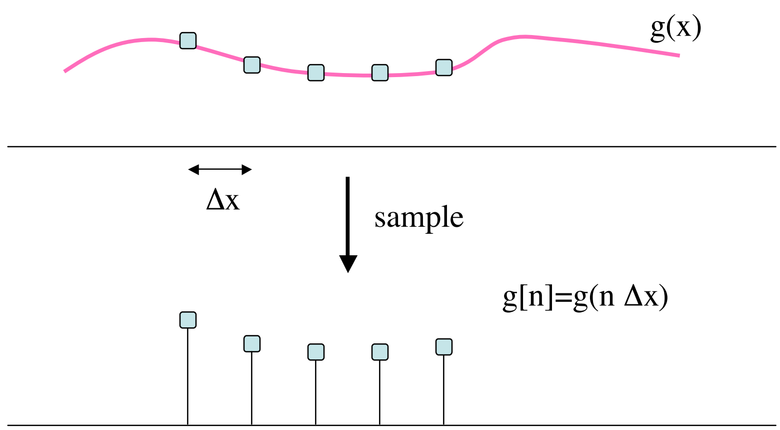


Analog vs. Digital

The Analog World:
Continuous time/space, continuous valued signals or images, e.g. vinyl records, photographs, x-ray films.

The Digital World:
Discrete time/space, discrete-valued signals or images, e.g. CD-Roms, DVDs, digital photos, digital x-rays, CT, MRI, ultrasound.

The Process of Sampling



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Questions

How finely do we need to sample?

What happens if we don’t sample finely enough?

Can we reconstruct the original signal or image from its samples?

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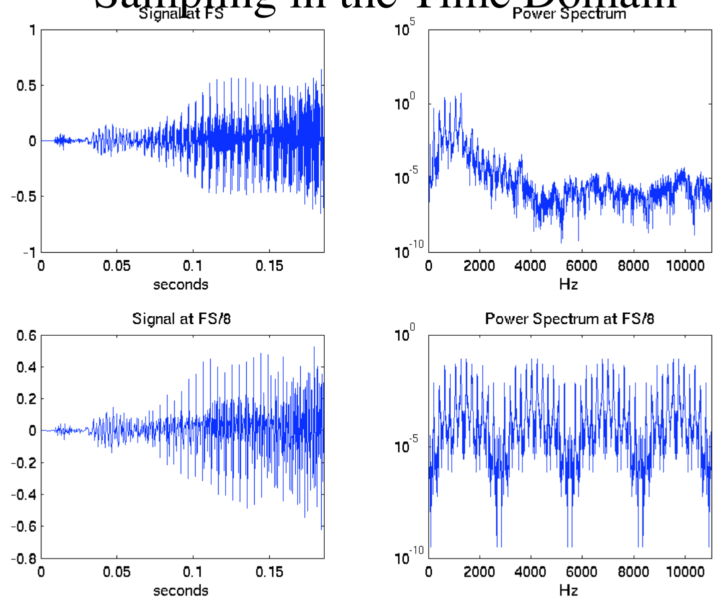
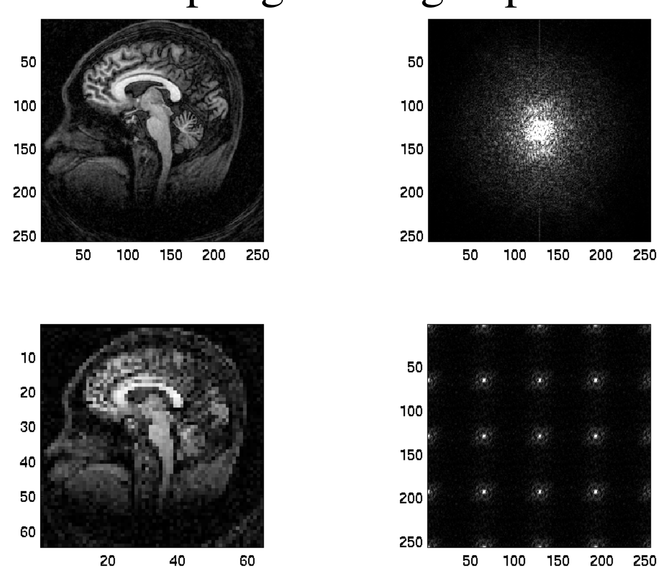
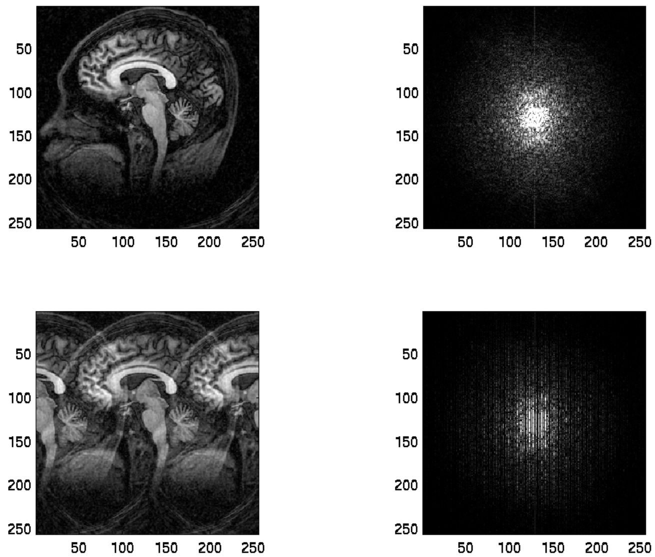
Sampling in the Time Domain



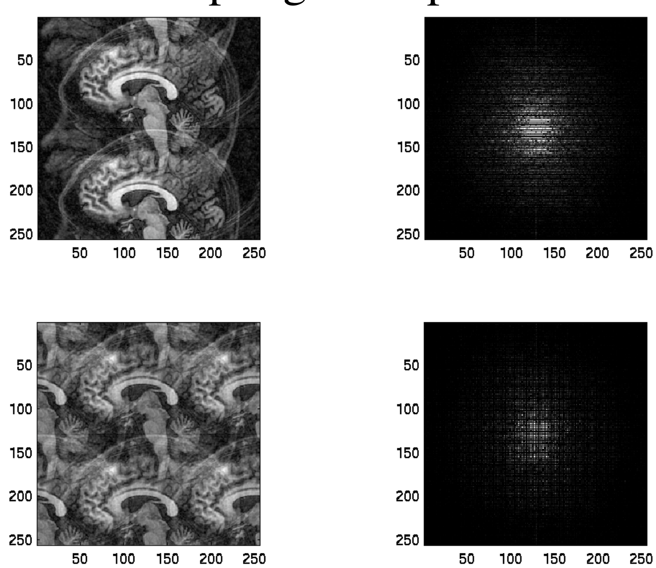
Sampling in Image Space



Sampling in k-space



Sampling in k-space



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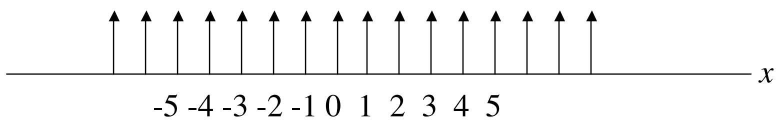
Comb Function

$$comb(x) = \sum_{n=-\infty}^{\infty} \delta(x - n)$$



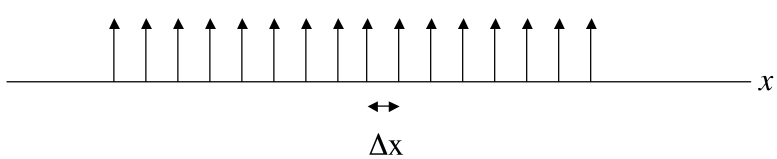
Other names: Impulse train, bed of nails, shah function.

Scaled Comb Function

$$comb\left(\frac{x}{\Delta x}\right) = \sum_{n=-\infty}^{\infty} \delta\left(\frac{x}{\Delta x} - n\right)$$

$$= \sum_{n=-\infty}^{\infty} \delta\left(\frac{x - n\Delta x}{\Delta x}\right)$$

$$= \Delta x \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x)$$



1D spatial sampling

$$g_S(x) = g(x)\frac{1}{\Delta x} comb\left(\frac{x}{\Delta x}\right)$$

$$= g(x) \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x)$$

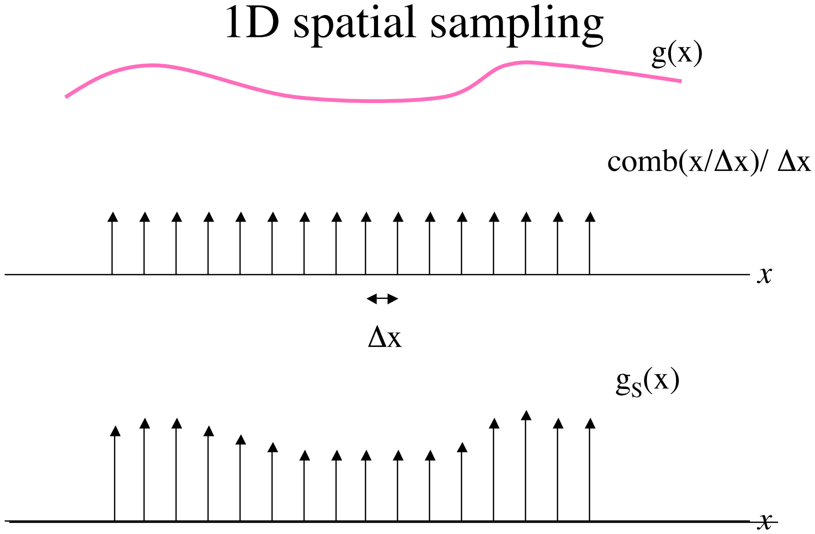
$$= \sum_{n=-\infty}^{\infty} g(n\Delta x)\delta(x - n\Delta x)$$

Recall the sifting property $\int_{-\infty}^{\infty} g(x)\delta(x-a) = g(a)$

But we can also write $\int_{-\infty}^{\infty} g(a)\delta(x-a) = g(a)\int_{-\infty}^{\infty} \delta(x-a) = g(a)$

So, $g(x)\delta(x-a) = g(a)\delta(x-a)$

1D spatial sampling



Fourier Transform of comb(x)

$$F\left[comb(x)\right] = comb(k_x)$$

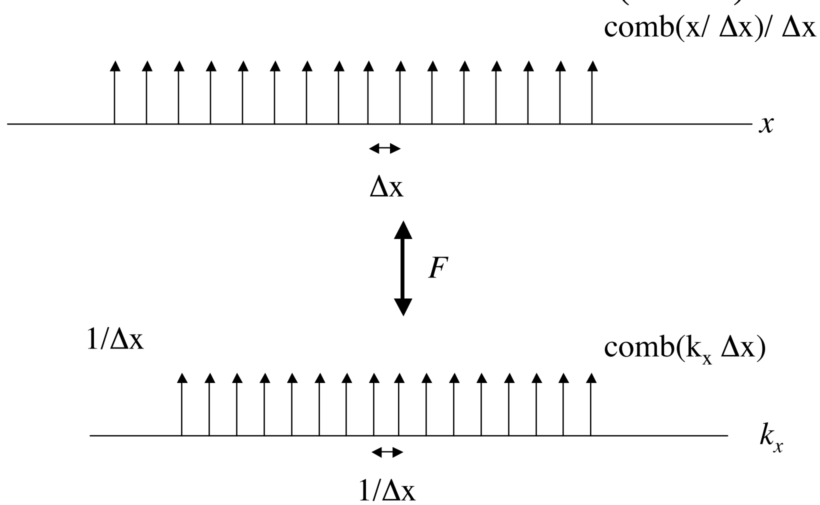
$$= \sum_{n=-\infty}^{\infty} \delta(k_x - n)$$

$$F\left[\frac{1}{\Delta x} comb(\frac{x}{\Delta x})\right] = \frac{1}{\Delta x} \Delta x comb(k_x \Delta x)$$

$$= \sum_{n=-\infty}^{\infty} \delta(k_x \Delta x - n)$$

$$= \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} \delta(k_x - \frac{n}{\Delta x})$$

Fourier Transform of comb(x/ Δx)



Fourier Transform of $g_S(x)$

$$F\left[g_S(x)\right] = F\left[g(x)\frac{1}{\Delta x}comb\left(\frac{x}{\Delta x}\right)\right]$$

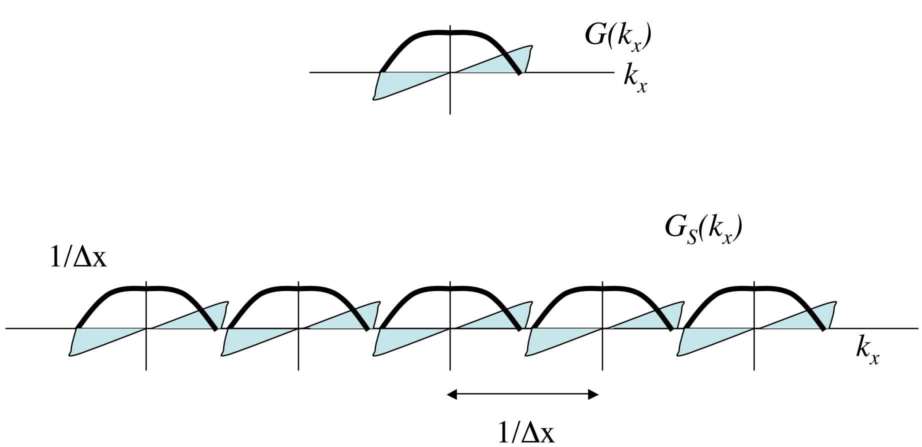
$$= G(k_x) * F\left[\frac{1}{\Delta x}comb\left(\frac{x}{\Delta x}\right)\right]$$

$$= G(k_x) * \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty}\delta\left(k_x - \frac{n}{\Delta x}\right)$$

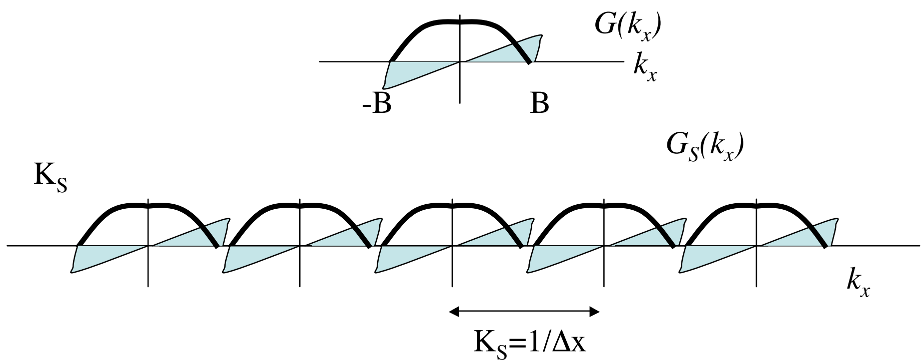
$$= \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty}G(k_x) * \delta\left(k_x - \frac{n}{\Delta x}\right)$$

$$= \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty}G\left(k_x - \frac{n}{\Delta x}\right)$$

Fourier Transform of $g_S(x)$



Nyquist Condition



To avoid overlap, we require that $1/\Delta x > 2B$ or
$K_S > 2B$ where $K_S = 1/\Delta x$ is the sampling frequency

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Example

Assume that the highest spatial frequency in an object is $B = 2\ \text{cm}^{-1}$.
Thus, smallest spatial period is 0.5 cm.

Nyquist theorem says we need to sample with
$\Delta x < 1/2B = 0.25$ cm

This corresponds to 2 samples per spatial period.

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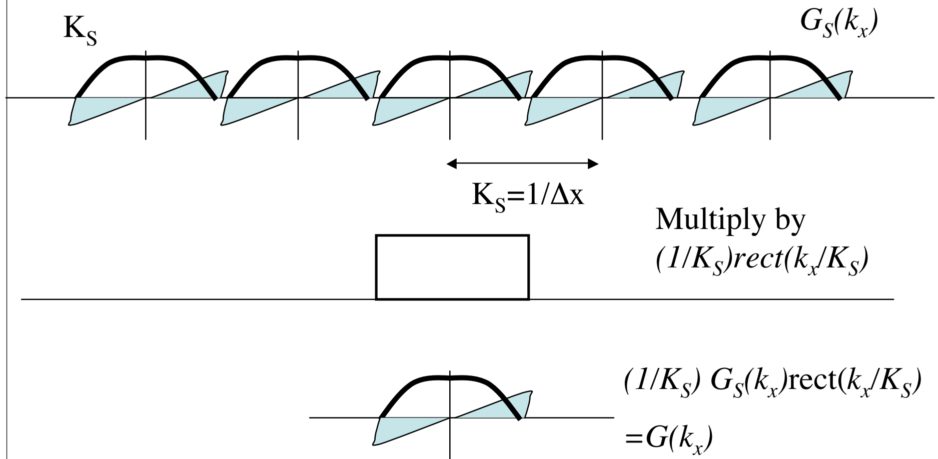
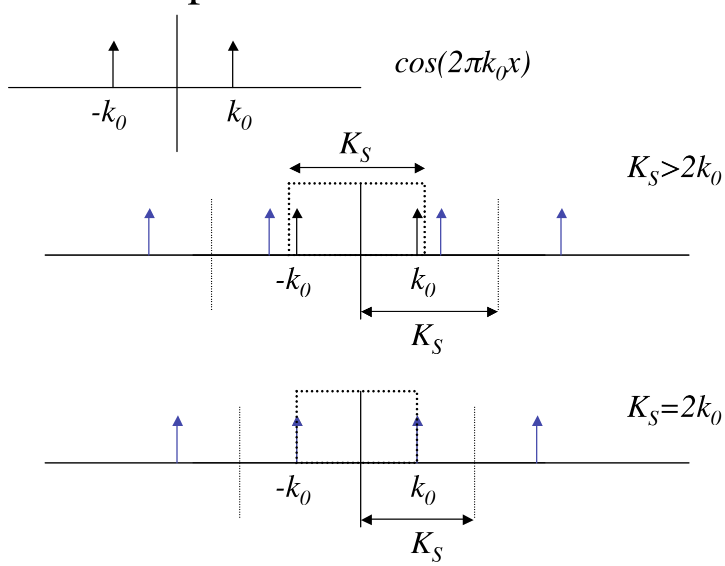
Reconstruction from Samples



Example Cosine Reconstruction



Reconstruction from Samples

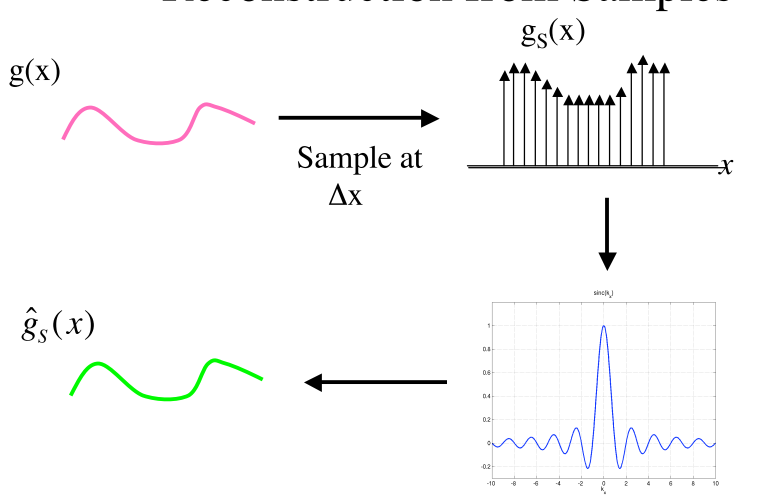
If the Nyquist condition is met, then

$$\hat{G}_S(k_x) = \frac{1}{K_S} G_S(k_x) rect(k_x / K_S) = G(k_x)$$

And the signal can be reconstructed by convolving the sample with a sinc function

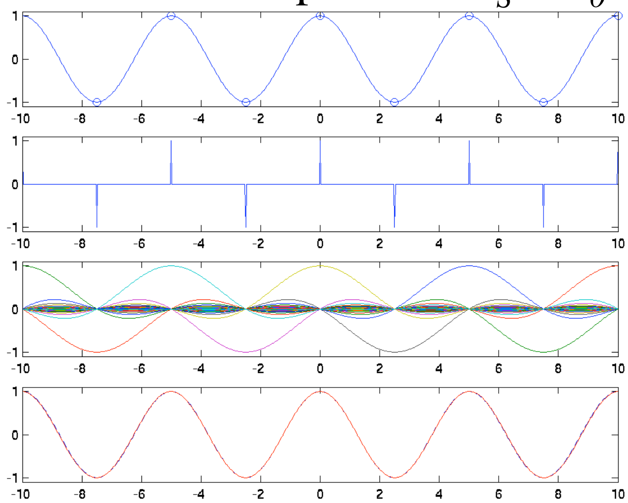
$$\begin{aligned}\hat{g}_S(x) &= g_S(x) * \text{sinc}(K_s x) \\ &= \left(\sum_{n=-\infty}^{\infty} g(n\Delta X)\delta(x - n\Delta X) \right) * \text{sinc}(K_s x) \\ &= \sum_{n=-\infty}^{\infty} g(n\Delta X)\, \text{sinc}(K_s(x - n\Delta x))\end{aligned}$$

Reconstruction from Samples



$$\text{sinc}(K_s x) = \text{sinc}(x / \Delta x)$$

Cosine Example with $K_S=2k_0$



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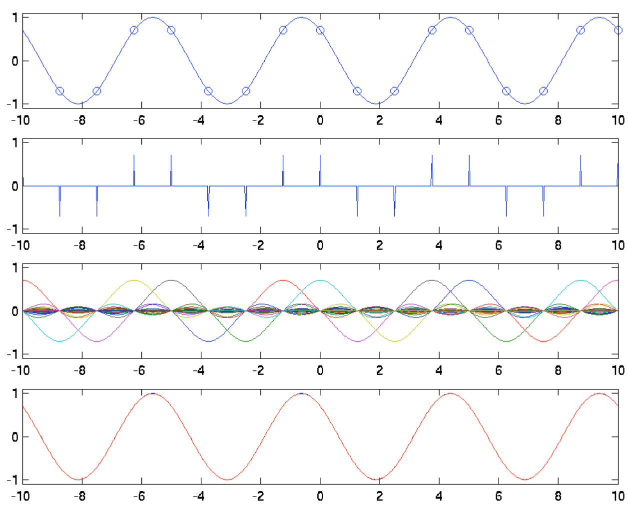
Example with $K_s=4k_0$



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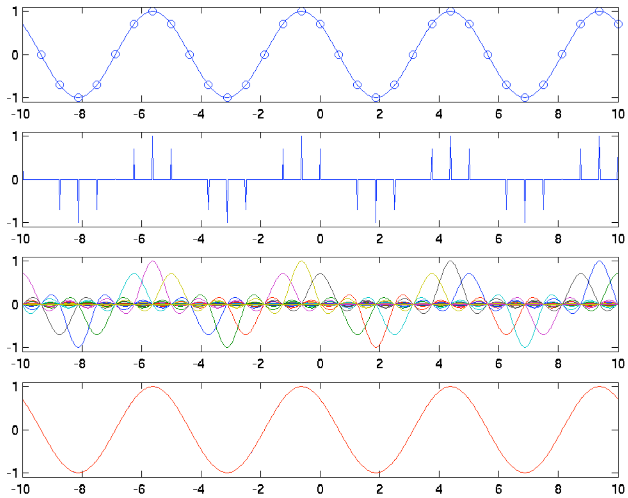
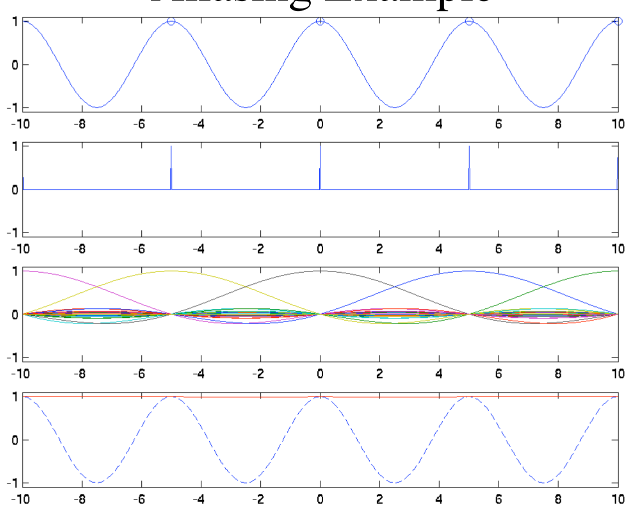
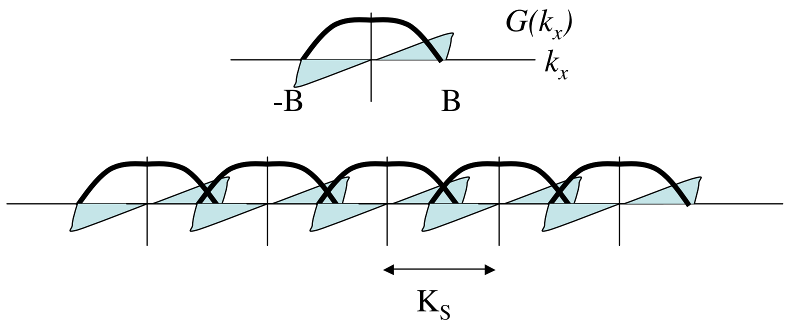
Example with $K_s=8k_0$



Aliasing Example



Aliasing



Aliasing occurs when the Nyquist condition is not satisfied.
This occurs for $K_S \leq 2B$

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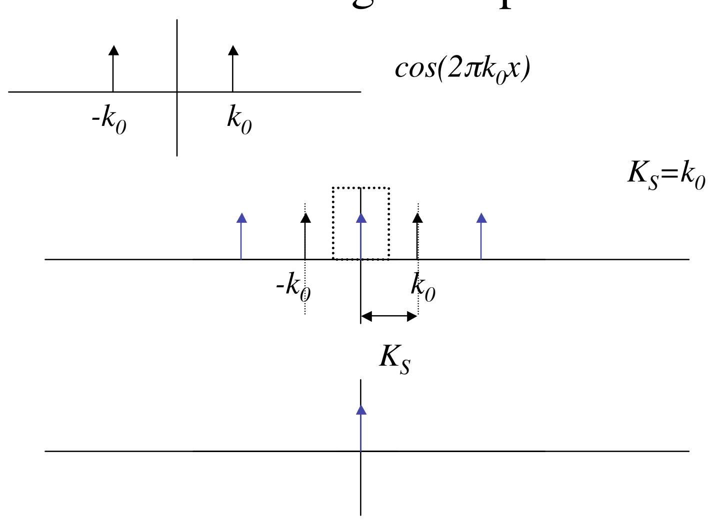
Aliasing Example



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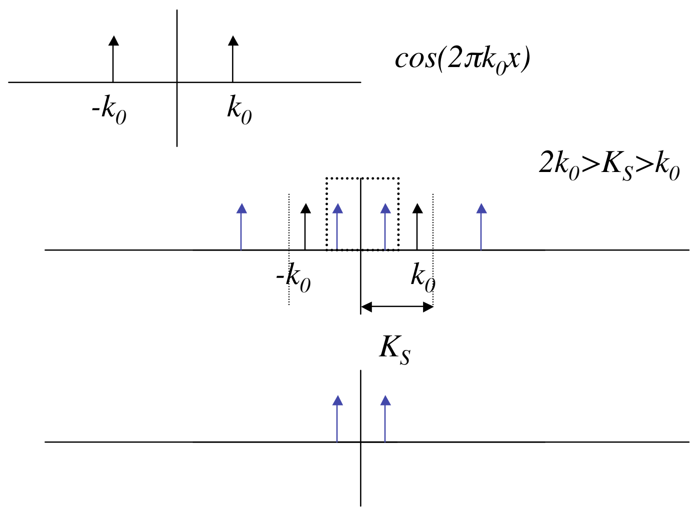
Aliasing Example



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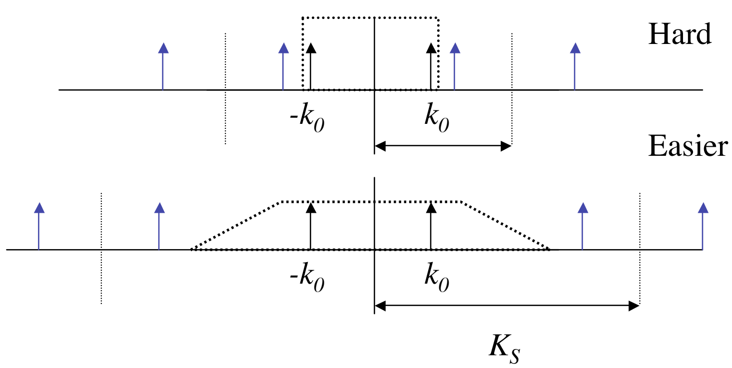
Practical Considerations

Why sample higher than the Nyquist frequency?

-- true sinc interpolation is not practical since the sinc function goes from -infinity to infinity

-- the requirements on the low-pass filter are reduced.



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