Bioengineering 280A
Principles of Biomedical Imaging

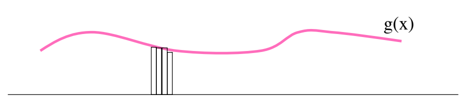
Fall Quarter 2005
Linear Systems Lecture 2

Representation of 1D Function

From the sifting property, we can write a 1D function as

$g(x)=\int_{-\infty}^{\infty} g(\xi)\delta(x-\xi)d\xi$. To gain intuition, consider the approximation

$g(x)\approx \sum_{n=-\infty}^{\infty} g(n\Delta x)\frac{1}{\Delta x}\Pi\left(\frac{x-n\Delta x}{\Delta x}\right)\Delta x.$



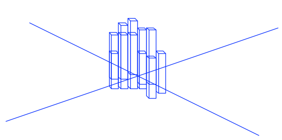
Representation of 2D Function

Similarly, we can write a 2D function as

$g(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x-\xi,y-\eta)d\xi d\eta.$

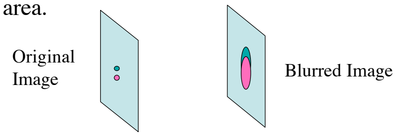
To gain intuition, consider the approximation

$g(x,y)\approx \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} g(n\Delta x,m\Delta y)\frac{1}{\Delta x}\Pi\left(\frac{x-n\Delta x}{\Delta x}\right)\frac{1}{\Delta y}\Pi\left(\frac{y-m\Delta y}{\Delta y}\right)\Delta x\Delta y.$



Impulse Response

Intuition: the impulse response is the response of a system to an input of infinitesimal width and unit area.



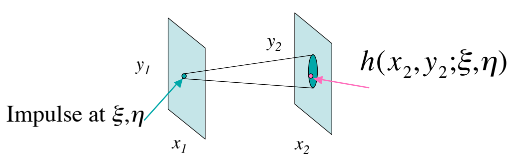
Since any input can be thought of as the weighted sum of impulses, a linear system is characterized by its impulse response(s).

Impulse Response

The impulse response characterizes the response of a system over all space to a Dirac delta impulse function at a certain location.

$h(x_2;\xi) = L[\delta(x_1 - \xi)]$ 1D Impulse Response

$h(x_2,y_2;\xi,\eta) = L[\delta(x_1 - \xi, y_1 - \eta)]$ 2D Impulse Response



Superposition Integral

What is the response to an arbitrary function $g(x_1,y_1)$?

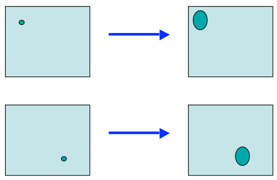
Write $g(x_1,y_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x_1 - \xi, y_1 - \eta)d\xi d\eta.$

The response is given by

$$\begin{aligned}
I(x_2,y_2) &= L[g_1(x_1,y_1)] \\
&= L\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x_1 - \xi, y_1 - \eta)d\xi d\eta\right] \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)L[\delta(x_1 - \xi, y_1 - \eta)]d\xi d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)h(x_2,y_2;\xi,\eta)d\xi d\eta
\end{aligned}$$

Space Invariance

If a system is space invariant, the impulse response depends only on the difference between the output coordinates and the position of the impulse and is given by $h(x_2, y_2; \xi, \eta) = h(x_2 - \xi, y_2 - \eta)$



2D Convolution

For a space invariant linear system, the superposition integral becomes a convolution integral.

$$
\begin{aligned}
I(x_2, y_2) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta) h(x_2, y_2; \xi, \eta) d\xi d\eta \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta) h(x_2 - \xi, y_2 - \eta) d\xi d\eta \\
&= g(x_2, y_2) ** h(x_2, y_2)
\end{aligned}
$$

where ** denotes 2D convolution. This will sometimes be abbreviated as *, e.g. $I(x_2, y_2) = g(x_2, y_2) * h(x_2, y_2)$.

1D Convolution

For completeness, here is the 1D version.

$$
\begin{aligned}
I(x) &= \int_{-\infty}^{\infty} g(\xi) h(x; \xi) d\xi \\
&= \int_{-\infty}^{\infty} g(\xi) h(x - \xi) d\xi \\
&= g(x) * h(x)
\end{aligned}
$$

Convolution with Dirac delta function

$$g(x)*\delta(x)=\int_{-\infty}^{\infty}g(\xi)\delta(x-\xi)d\xi = g(x)$$

$$g(x)*\delta(x-\Delta)=\int_{-\infty}^{\infty}g(\xi)\delta(x-\Delta-\xi)d\xi = g(x-\Delta)$$

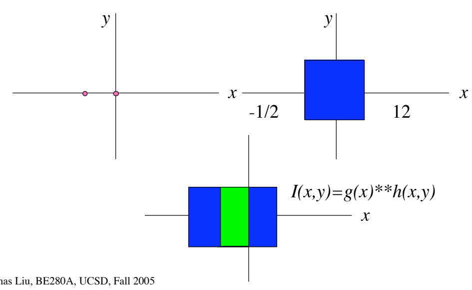
Convolution of g(x) with a shifted Dirac delta function just shifts g(x)

$$g(x,y)**\delta(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(\xi,\eta)\delta(x-\xi,y-\eta)d\xi d\eta = g(x,y)$$

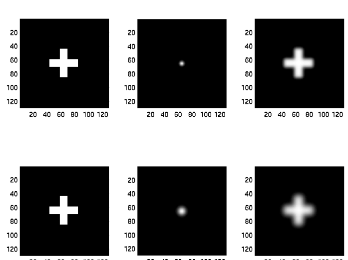
$$g(x,y)**\delta(x-x_0,y-y_0)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}g(\xi,\eta)\delta(x-x_0-\xi,y-y_0-\eta)d\xi d\eta = g(x-x_0,y-y_0)$$

2D Convolution Example

g(x)= δ(x+1/2,y) + δ(x,y) *h(x)=rect(x,y)*



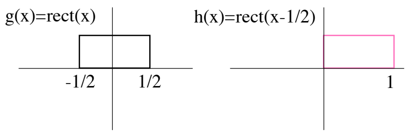
2D Convolution Example



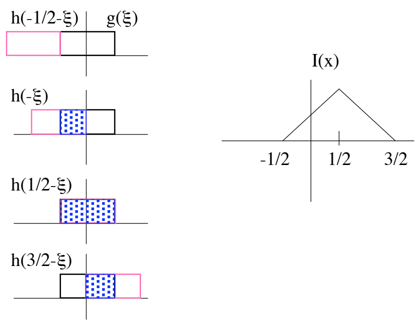
1D Convolution Review

$$g(x) * h(x) = \int_{-\infty}^{\infty} g(\xi) h(x-\xi) d\xi$$

Basic Rule: Flip one function, slide it past the other function, and integrate as you go.



1D Convolution Review



Convolution/Modulation Theorem

$$\begin{aligned} F\{g(x) * h(x)\} &= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} g(u) * h(x-u) du \right] e^{-j2\pi k_x x} dx \\ &= \int_{-\infty}^{\infty} g(u) \int_{-\infty}^{\infty} h(x-u) e^{-j2\pi k_x x} dx du \\ &= \int_{-\infty}^{\infty} g(u) H(k_x) e^{-j2\pi k_x u} du \\ &= G(k_x) H(k_x) \end{aligned}$$

Convolution in the spatial domain transforms into multiplication in the frequency domain. Dual is modulation

$$F\{g(x)h(x)\} = G(k_x) * H(k_x)$$

2D Convolution/Multiplication

Convolution

$$F\left[g(x,y)**h(x,y)\right]=G(k_x,k_y)H(k_x,k_y)$$

Multiplication

$$F\left[g(x,y)h(x,y)\right]=G(k_x,k_y)**H(k_x,k_y)$$

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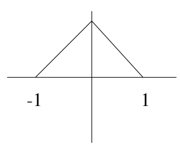
Application of Convolution Thm.

$$\Lambda(x)=\begin{cases}1-|x| & |x|<1\\ 0 & otherwise\end{cases}$$

$$F(\Lambda(x))=\int_{-1}^{1}\left(1-|x|\right)e^{-j2\pi k_x x}dx=??$$



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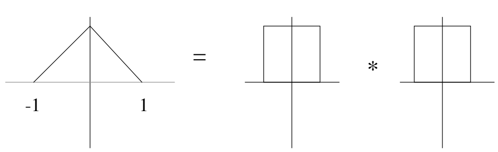
Application of Convolution Thm.

$$\Lambda(x)=\Pi(x)*\Pi(x)$$

$$F(\Lambda(x))=\mathrm{sin}c^2\left(k_x\right)$$



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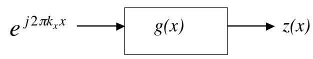
Eigenfunctions

The fundamental nature of the convolution theorem may be better understood by observing that the complex exponentials are eigenfunctions of the convolution operator.

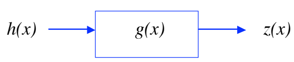


$$\begin{aligned} z(x) &= g(x) * e^{j2\pi k_x x} \\ &= \int_{-\infty}^{\infty} g(u) e^{j2\pi k_x (x-u)} du \\ &= G(k_x) e^{j2\pi k_x x} \end{aligned}$$

The response of a linear shift invariant system to a complex exponential is simply the exponential multiplied by the FT of the system's impulse response.

Eigenfunctions

Now consider an arbitrary input *h(x)*.



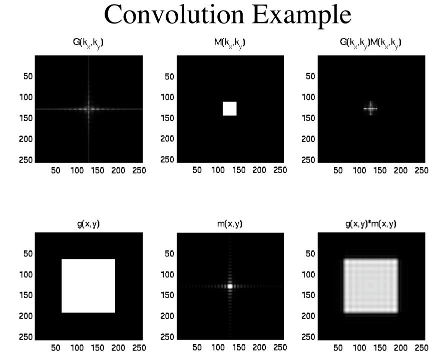
Recall that we can express *h(x)* as the integral of weighted complex exponentials.

$$h(x) = \int_{-\infty}^{\infty} H(k_x) e^{j2\pi k_x x} dk_x$$

Each of these exponentials is weighted by $G(k_x)$ so that the response may be written as

$$z(x) = \int_{-\infty}^{\infty} G(k_x) H(k_x) e^{j2\pi k_x x} dk_x$$

Convolution Example

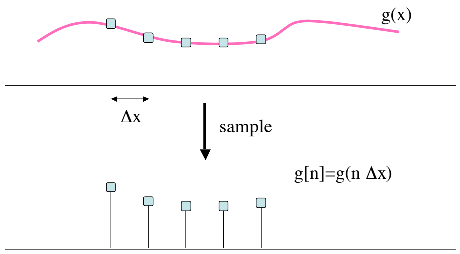


Analog vs. Digital

The Analog World:
Continuous time/space, continuous valued signals or images, e.g. vinyl records, photographs, x-ray films.

The Digital World:
Discrete time/space, discrete-valued signals or images, e.g. CD-Roms, DVDs, digital photos, digital x-rays, CT, MRI, ultrasound.

The Process of Sampling



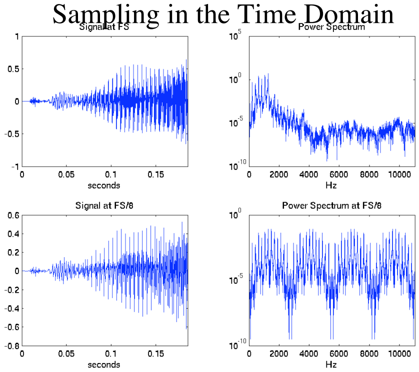
Questions

How finely do we need to sample?

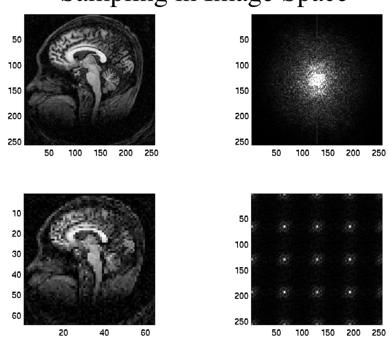
What happens if we don't sample finely enough?

Can we reconstruct the original signal or image from its samples?

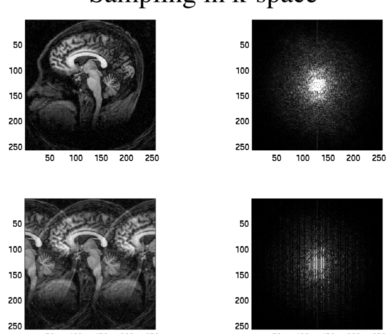
Sampling in the Time Domain



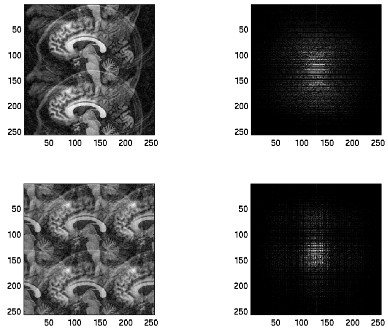
Sampling in Image Space



Sampling in k-space

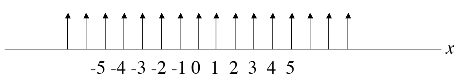


Sampling in k-space



Comb Function

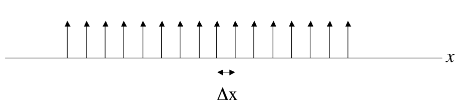
$$comb(x) = \sum_{n=-\infty}^{\infty} \delta(x-n)$$



Other names: Impulse train, bed of nails, shah function.

Scaled Comb Function

$$comb\left(\frac{x}{\Delta x}\right) = \sum_{n=-\infty}^{\infty} \delta\left(\frac{x}{\Delta x} - n\right)$$
$$= \sum_{n=-\infty}^{\infty} \delta\left(\frac{x - n\Delta x}{\Delta x}\right)$$
$$= \Delta x \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x)$$



1D spatial sampling

$$g_S(x) = g(x)\frac{1}{\Delta x}comb\left(\frac{x}{\Delta x}\right)$$

$$= g(x)\sum_{n=-\infty}^{\infty}\delta(x - n\Delta x)$$

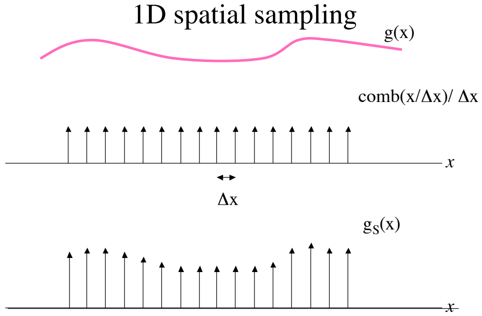
$$= \sum_{n=-\infty}^{\infty} g(n\Delta x)\delta(x - n\Delta x)$$

Recall the sifting property $\int_{-\infty}^{\infty} g(x)\delta(x-a) = g(a)$

But we can also write $\int_{-\infty}^{\infty} g(a)\delta(x-a) = g(a)\int_{-\infty}^{\infty}\delta(x-a) = g(a)$

So, $g(x)\delta(x-a) = g(a)\delta(x-a)$

1D spatial sampling



Fourier Transform of comb(x)

$$F\left[comb(x)\right] = comb(k_x)$$

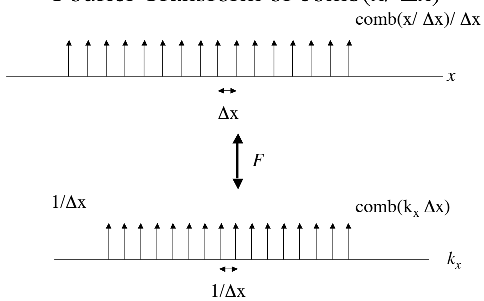
$$= \sum_{n=-\infty}^{\infty}\delta(k_x - n)$$

$$F\left[\frac{1}{\Delta x}comb(\frac{x}{\Delta x})\right] = \frac{1}{\Delta x}\Delta x comb(k_x \Delta x)$$

$$= \sum_{n=-\infty}^{\infty}\delta(k_x\Delta x - n)$$

$$= \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty}\delta(k_x - \frac{n}{\Delta x})$$

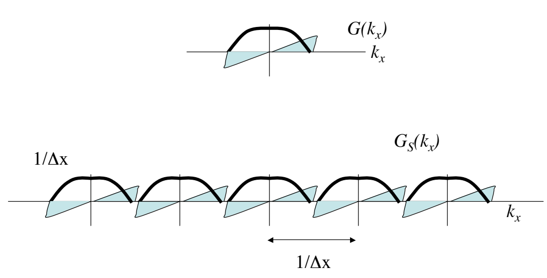
Fourier Transform of comb(x/ Δx)



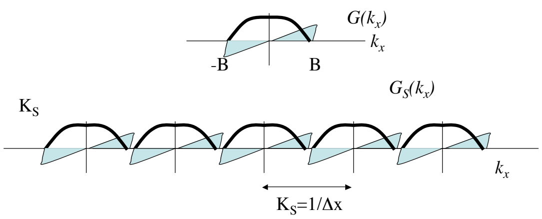
Fourier Transform of $g_S(x)$

$$
\begin{aligned}
F\left[g_S(x)\right] &= F\left[g(x)\frac{1}{\Delta x}comb\left(\frac{x}{\Delta x}\right)\right] \\
&= G(k_x) * F\left[\frac{1}{\Delta x}comb\left(\frac{x}{\Delta x}\right)\right] \\
&= G(k_x) * \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty}\delta\left(k_x - \frac{n}{\Delta x}\right) \\
&= \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty}G(k_x) * \delta\left(k_x - \frac{n}{\Delta x}\right) \\
&= \frac{1}{\Delta x}\sum_{n=-\infty}^{\infty}G\left(k_x - \frac{n}{\Delta x}\right)
\end{aligned}
$$

Fourier Transform of $g_S(x)$



Nyquist Condition



To avoid overlap, we require that $1/\Delta x > 2B$ or
$K_S > 2B$ where $K_S = 1/\Delta x$ is the sampling frequency

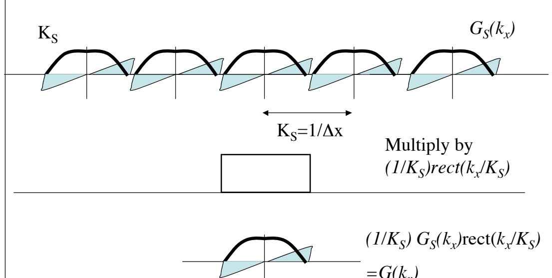
Example

Assume that the highest spatial frequency in an object is $B = 2\ \text{cm}^{-1}$.
Thus, smallest spatial period is 0.5 cm.

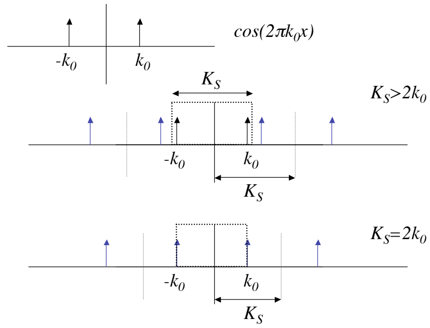
Nyquist theorem says we need to sample with
$\Delta x < 1/2B = 0.25$ cm

This corresponds to 2 samples per spatial period.

Reconstruction from Samples



Example Cosine Reconstruction



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Reconstruction from Samples

If the Nyquist condition is met, then

$$\hat{G}_S(k_x) = \frac{1}{K_S} G_S(k_x) rect(k_x / K_S) = G(k_x)$$

And the signal can be reconstructed by convolving the sample with a sinc function

$$\begin{aligned}\hat{g}_S(x) &= g_S(x) * \mathrm{sinc}(K_s x) \\ &= \left(\sum_{n=-\infty}^{\infty} g(n\Delta X)\delta(x - n\Delta X)\right) * \mathrm{sinc}(K_s x) \\ &= \sum_{n=-\infty}^{\infty} g(n\Delta X)\,\mathrm{sinc}(K_s(x - n\Delta x))\end{aligned}$$

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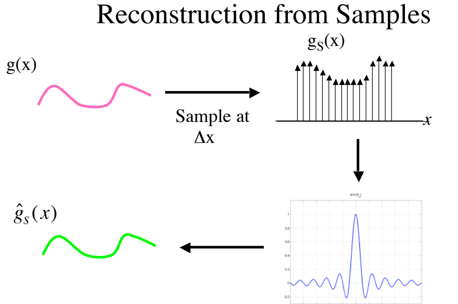
Reconstruction from Samples



$\mathrm{sinc}(K_s x) = \mathrm{sinc}(x / \Delta x)$

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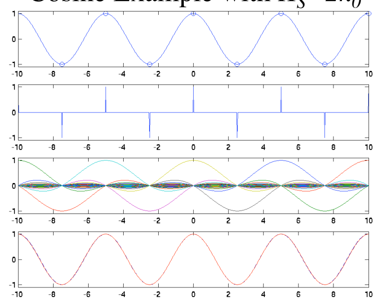
Cosine Example with $K_S=2k_0$



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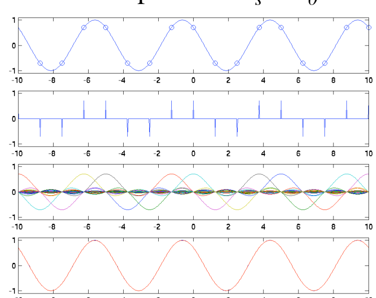
Example with $K_s=4k_0$



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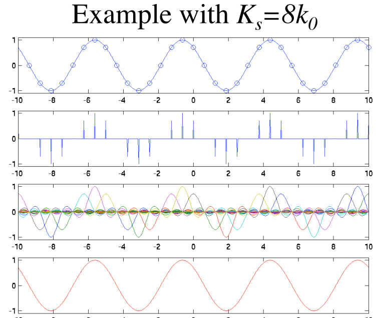
Example with $K_s=8k_0$



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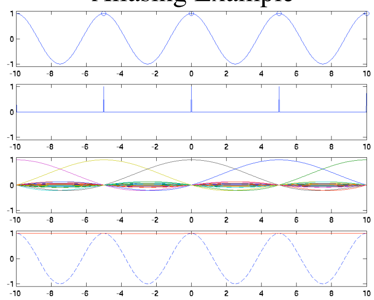
Aliasing Example



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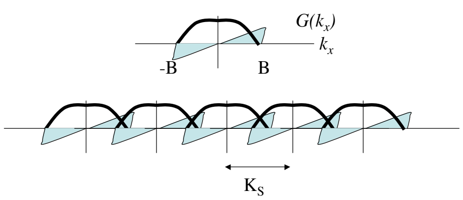
Aliasing



Aliasing occurs when the Nyquist condition is not satisfied. This occurs for $K_S \leq 2B$

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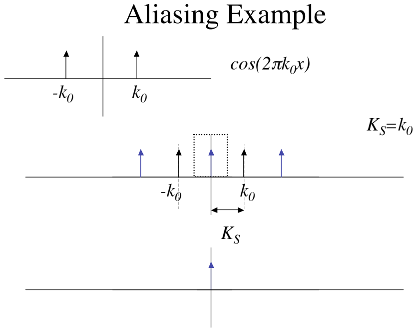
Aliasing Example



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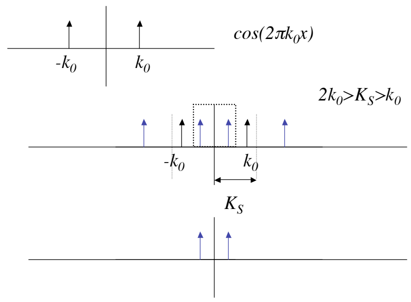
Aliasing Example



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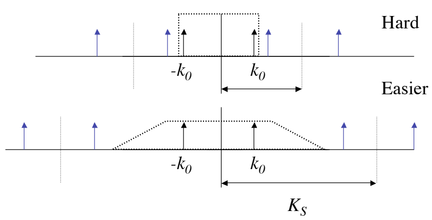
Practical Considerations

Why sample higher than the Nyquist frequency?

-- true sinc interpolation is not practical since the sinc function goes from -infinity to infinity

-- the requirements on the low-pass filter are reduced.



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