Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2007
CT/Fourier Lecture 2

Topics

• Projection Slice Theorem
• Fourier Transforms

Projection Slice Theorem

\[ G(\rho, \theta) = \int_{-\infty}^{\infty} g(l, \theta) e^{-j2\pi \rho l} dl \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \theta + y \sin \theta - l) e^{-j2\pi \rho l} dx dy \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-j2\pi \rho (x \cos \theta + y \sin \theta)} dx dy \]

\[ = F_{2D}[f(x, y)]\big|_{\rho, \phi = \rho, \theta} \]

The Fourier Transform

Fourier Transform (FT)

\[ G(f) = \int_{-\infty}^{\infty} g(t) e^{-j2\pi \phi} dt = F\{g(t)\} \]

Inverse Fourier Transform

\[ g(t) = \int_{-\infty}^{\infty} G(f) e^{j2\pi \phi} df = F^{-1}\{G(f)\} \]
**Units**

Temporal Coordinates, e.g. $t$ in seconds, $f$ in cycles/second

\[ G(f) = \int_{-\infty}^{\infty} g(t)e^{-2\pi if}dt \] Fourier Transform

\[ g(t) = \int_{-\infty}^{\infty} G(f)e^{2\pi if}df \] Inverse Fourier Transform

Spatial Coordinates, e.g. $x$ in cm, $k_x$ is spatial frequency in cycles/cm

\[ G(k_x) = \int_{-\infty}^{\infty} g(x)e^{-2\pi ik_x}dx \] Fourier Transform

\[ g(x) = \int_{-\infty}^{\infty} G(k_x)e^{2\pi ik_x}dk_x \] Inverse Fourier Transform

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**2D Fourier Transform**

Fourier Transform

\[ G(k_x,k_y) = F \left\{ g(x,y) \right\} = \iint_{-\infty}^{\infty} g(x,y)e^{-j2\pi(k_x x + k_y y)}dxdy \]

Inverse Fourier Transform

\[ g(x,y) = \iiint_{-\infty}^{\infty} G(k_x,k_y)e^{j2\pi(k_x x + k_y y)}dk_xdk_y \]

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**1D Fourier Transform**

**Plane Waves**

\[ e^{j2\pi(k_x x + k_y y)} = \cos(2\pi(k_x x + k_y y)) + j\sin(2\pi(k_x x + k_y y)) \]

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**Units**

**2D Fourier Transform**

**1D Fourier Transform**

**Plane Waves**
Figure 2.5 from Prince and Link

Plane Waves

\[
\frac{1}{k_x} x + \frac{1}{k_y} y = \frac{1}{k_x} x_2 + \frac{1}{k_y} y_2
\]

\[
\theta = \arctan \left( \frac{k_y}{k_x} \right)
\]

k-space

Image space

k-space

Fourier Transform

Examples
Computing Transforms

\[ F(\delta(x)) = \int_{-\infty}^{\infty} \delta(x)e^{-j2\pi kx} \, dx = 1 \]

\[ F(\delta(x-x_0)) = \int_{-\infty}^{\infty} \delta(x-x_0)e^{-j2\pi kx} \, dx = e^{-j2\pi kx_0} \]

\[ F(\Pi(x)) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi kx} \, dx \]
\[ = e^{j\pi k} - e^{-j\pi k} \]
\[ = -j2\pi k \]
\[ = \frac{\sin(\pi k)}{\pi k} = \text{sinc}(k) \]

Define \( h(k_x) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} \, dx \) and see what it does under an integral.

\[ \int_{-\infty}^{\infty} G(k_x)h(k_x)dk_x = \int_{-\infty}^{\infty} G(k_x)e^{-j2\pi k_x x}dk_x \]
\[ = \int_{-\infty}^{\infty} G(k_x)e^{-j2\pi k_x x}dk_x \]
\[ = \int_{\infty}^{\infty} g(-x)dx \]
\[ = G(0) \]

Therefore, \( F(1) = \int_{-\infty}^{\infty} e^{-j2\pi k_x x} \, dx = \delta(k_x) \)
Computing Transforms

Similarly,

\[ F\{e^{j2\pi k_0 x}\} = \delta(k_x - k_0) \]
\[ F\{\cos 2\pi k_0 x\} = \frac{1}{2}(\delta(k_x - k_0) + \delta(k_x + k_0)) \]
\[ F\{\sin 2\pi k_0 x\} = \frac{1}{2j}(\delta(k_x - k_0) - \delta(k_x + k_0)) \]

Examples

\[ g(x, y) = 1 + e^{j2\pi(ax + by)} \]
\[ G(k_x, k_y) = \delta(k_x, k_y) + \delta(k_x + a)\delta(k_y) + \delta(k_x - a)\delta(k_y) \]

\[ g(x, y) = \cos(2\pi(ax + by)) \]
\[ G(k_x, k_y) = \frac{1}{2}\delta(k_x - a)\delta(k_y - b) + \frac{1}{2}\delta(k_x + a)\delta(k_y + b) \]

Examples

\[ G(k_x, k_y) = \delta(k_x, k_y) + \delta(k_x + c)\delta(k_y) + \delta(k_x)\delta(k_y - d) + \frac{1}{2}\delta(k_x - a)\delta(k_y - b) + \frac{1}{2}\delta(k_x + a)\delta(k_y + b) \]
\[ g(x, y) = ??? \]
**Basic Properties**

**Linearity**

\[ F \{ ag(x,y) + bh(x,y) \} = aG(k_1,k_2) + bH(k_1,k_2) \]

**Scaling**

\[ F \{ g(ax,by) \} = \frac{1}{|ab|} G \left( \frac{k_1}{a}, \frac{k_2}{b} \right) \]

**Shift**

\[ F \{ g(x-\alpha, y-\beta) \} = G(k_1, k_2) e^{-j2\pi(k_1x+k_2y)} \]

**Modulation**

\[ F \{ g(x,y) e^{j2\pi(ax+by)} \} = G(k_1 - a, k_2 - b) \]

**Scaling Theorem**

\[ F \{ g(ax) \} = \frac{1}{|a|} G \left( \frac{k_1}{a} \right) \]
\[ F \{ g(ax,by) \} = \frac{1}{|ab|} G \left( \frac{k_1}{a}, \frac{k_2}{b} \right) \]

**Linearity**

The Fourier Transform is linear.

\[ F \{ ag(x) + bh(x) \} = aG(k_1) + bH(k_1) \]

**Separable Functions**

\[ g(x,y) \] is said to be a separable function if it can be written as \[ g(x,y) = g_x(x)g_y(y) \]

The Fourier Transform is then separable as well.

\[ G(k_1,k_2) = \int_{-\infty}^{\infty} g(x,y) e^{-j2\pi(k_1x+k_2y)} \, dx \, dy \]
\[ = \int_{-\infty}^{\infty} g_x(x) e^{-j2\pi k_1x} \, dx \int_{-\infty}^{\infty} g_y(y) e^{-j2\pi k_2y} \, dy \]
\[ = G_x(k_1)G_y(k_2) \]

**Example**

\[ g(x,y) = \Pi(x)\Pi(y) \]
\[ G(k_1,k_2) = \text{sinc}(k_1)\text{sinc}(k_2) \]
Example (sinc/rect)

\[
g(x, y) = \Pi(x)\Pi(y)
\]

\[
G(k_x, k_y) = \text{sinc}(k_x)\text{sinc}(k_y)
\]

Examples

\[
g(x, y) = \delta(x, y) = \delta(x)\delta(y)
\]

\[
G(k_x, k_y) = 1
\]

\[
g(x, y) = \delta(x)
\]

\[
G(k_x, k_y) = \delta(k_y)
\]

Duality

Note the similarity between these two transforms

\[
F\left\{e^{j2\pi x^2}\right\} = \delta(k_x - a)
\]

\[
F\left\{\delta(x - a)\right\} = e^{-j2\pi k_x a}
\]

These are specific cases of duality

\[
F\{G(x)\} = g(-k_x)
\]
Application of Duality

\[ F\{\text{sinc}(x)\} = \int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x} e^{-j2\pi kx} dx = ?? \]

Recall that \( F\{\Pi(x)\} = \text{sinc}(k) \).
Therefore from duality, \( F\{\text{sinc}(x)\} = \Pi(-k) = \Pi(k) \)

Shift Theorem

\[ F\{g(x - a)\} = G(k)e^{-j2\pi ak} \]
\[ F\{g(x - a,y - b)\} = G(k_x,k_y)e^{-j2\pi(k_xa+k_yb)} \]

Shifting the function doesn’t change its spectral content, so the magnitude of the transform is unchanged. Each frequency component is shifted by \( a \). This corresponds to a relative phase shift of
\[ -\frac{2\pi a}{\text{spatial period}} = -2\pi ak \]
For example, consider \( \exp(j2\pi k_x x) \). Shifting this by \( a \) yields \( \exp(j2\pi k_x (x - a)) = \exp(j2\pi k_x x)\exp(-j2\pi ak_x) \)