Bioengineering 280A  
Principles of Biomedical Imaging  
Fall Quarter 2007  
CT/Fourier Lecture 3

Topics

- Modulation Transfer Function
- Convolution/Multiplication
- Modulation
- Revisit Projection-Slice Theorem
- Filtered Backprojection

\[ e^{j2\pi kx} \rightarrow g(x) \rightarrow z(x) \]

\[ z(x) = g(x) \ast e^{j2\pi kx} \]
\[ = \int_{-\infty}^{\infty} g(u)e^{j2\pi k(x-u)} du \]
\[ = G(k)x e^{j2\pi kx} \]

The response of a linear shift invariant system to a complex exponential is simply the exponential multiplied by the FT of the system’s impulse response.

8. Referring to Figure 1 (above) which demonstrates 3 different line spread functions (LSFs), which LSF will yield the best spatial resolution?

10. Referring to Figure 1 which shows LSFs, and Figure 2 which shows the corresponding modulation transfer functions (MTFs), which MTF corresponds to LSF C?

A. MTF number 1  
B. MTF number 2  
C. MTF number 3

D14. The intrinsic resolution of a gamma camera is 5 mm. The collimator resolution is 10 mm. The overall system resolution is _______ mm.

A. 15  
B. 11.2  
C. 7.5  
D. 5.0  
E. 0.5
MTF = Fourier Transform of PSF

Modulation Transfer Function (MTF) or Frequency Response

Modulation Transfer Function
Convolution/Multiplication

Now consider an arbitrary input $h(x)$.

$$h(x) \xrightarrow{g(x)} z(x)$$

Recall that we can express $h(x)$ as the integral of weighted complex exponentials.

$$h(x) = \int_{-\infty}^{\infty} H(k) e^{j2\pi k x} dk$$

Each of these exponentials is weighted by $G(k)$ so that the response may be written as

$$z(x) = \int_{-\infty}^{\infty} G(k) H(k) e^{j2\pi k x} dk$$

2D Convolution/Multiplication

**Convolution**

$$F\left[ g(x,y) * h(x,y) \right] = G(k_x,k_y) H(k_x,k_y)$$

**Multiplication**

$$F\left[ g(x,y) h(x,y) \right] = G(k_x,k_y) * H(k_x,k_y)$$

Application of Convolution Thm.

$$A(x) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F(A(x)) = \int_{-1}^{1} (1-|x|) e^{-j2\pi k x} dx = ??$$
Application of Convolution Thm.

\[ \Lambda(x) = \Pi(x) \star \Pi(x) \]

\[ F(\Lambda(x)) = \sin^2(k_x) \]

Response of an Imaging System

\[ z(x,y) = g(x,y) \star h_1(x,y) \star h_2(x,y) \star h_3(x,y) \]

System MTF = Product of MTFs of Components

Bushberg et al 2001
Useful Approximation

\[
\text{FWHM}_{\text{System}} = \sqrt{\text{FWHM}_1^2 + \text{FWHM}_2^2 + \cdots + \text{FWHM}_N^2}.
\]

**Example**

\[
\begin{align*}
\text{FWHM}_1 &= 1 \text{mm} \\
\text{FWHM}_2 &= 2 \text{mm} \\
\text{FWHM}_{\text{System}} &= \sqrt{5} = 2.24 \text{mm}
\end{align*}
\]

**Modulation**

**Amplitude Modulation (e.g. AM Radio)**

\[
g(t) \rightarrow 2g(t) \cos(2\pi f_0 t)
\]

\[
G(f) \rightarrow G(f-f_0) + G(f+f_0)
\]

**Modulation Example**

\[
F[g(x)e^{j2\pi k_0 x}] = G(k_x) \ast \delta(k_x - k_0) = G(k_x - k_0)
\]

\[
F[g(x)\cos(2\pi k_0 x)] = \frac{1}{2} G(k_x - k_0) + \frac{1}{2} G(k_x + k_0)
\]

\[
F[g(x)\sin(2\pi k_0 x)] = \frac{1}{2j} G(k_x - k_0) - \frac{1}{2j} G(k_x + k_0)
\]
Projection Theorem

\[ U(k,0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(x,y)e^{-2\pi i (kx + ky)} \, dx \, dy \]

In-Class Example:
\[ \mu(x,y) = \cos 2\pi x \]

Projection Slice Theorem

\[ G(\rho,\theta) = \int_{-\infty}^{\infty} g(l,\theta)e^{-2\pi i \rho \theta} \, dl \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \delta(x \cos \theta + y \sin \theta - l) e^{-2\pi i \rho \theta} \, dx \, dy \, dl \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-2\pi i \rho (x \cos \theta + y \sin \theta)} \, dx \, dy \]

\[ = F_{2D}[f(x,y)]_{(x \cos \theta + y \sin \theta) = \rho \sin \theta} \]

Fourier Reconstruction

Interpolate onto Cartesian grid then take inverse transform
Polar Version of Inverse FT

\[
\mu(x, y) = \int_0^{2\pi} \int_{-\infty}^{\infty} G(k_x, k_y) e^{j(2\pi(k_x x + k_y y))} dk_x dk_y
\]

\[
= \int_0^{2\pi} \int_{-\infty}^{\infty} G(k_x, k_y) e^{j(2\pi(k_x x + k_y y))} dk_x dk_y
\]

Note:
\[g(l, \theta + \pi) = g(-l, \theta)\]
So
\[G(k_x, k_y + \pi) = G(-k_x, k_y)\]

Filtered Backprojection

\[
\mu(x, y) = \int_0^{2\pi} \int_{-\infty}^{\infty} G(k_x, k_y) e^{j(2\pi(k_x x + k_y y))} dk_x dk_y
\]

\[
= \int_0^{2\pi} \int_{-\infty}^{\infty} \|G(k_x, k_y)\| e^{j(2\pi l)} dk_x dk_y
\]

where \(l = x \cos \theta + y \sin \theta\)

\[
g'(l, \theta) = \int_{-\infty}^{\infty} \|G(k_x, k_y)\| e^{j(2\pi l)} dk_x dk_y
\]

\[
= g(l, \theta) * F^{-1}[k]
\]

\[
= g(l, \theta) * q(l)
\]

where \(l = x \cos \theta + y \sin \theta\)

Low frequencies are oversampled. So to compensate for this, multiply the k-space data by \(|k|\) before inverse transforming.

Fourier Interpretation

\[
\text{Density} = \frac{N}{\text{circumference}} = \frac{N}{2\pi|k|}
\]

Ram-Lak Filter

\[k_{\text{max}} = \frac{1}{\Delta s}\]
Reconstruction Path

Projection

F

F⁻¹

Filtered Projection

Back-Project

Example

Figure 6.15
Convolution steps:
(a) Original sinogram;
(b) filtered sinogram;
(c) profile of sinogram row [white line in (a)]; and
(d) profile of filtered sinogram row [white line in (b)].
Example

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Prince and Links 2005