Bioengineering 280A  
Principles of Biomedical Imaging  
Fall Quarter 2007  
CT/Fourier Lecture 4

**Topics**
- Sampling Requirements in CT  
- Sampling Theory  
- Aliasing

**CT Sampling Requirements**
What should the size of the detectors be?
How many detectors do we need?
How many views do we need?

**View Aliasing**

Kak and Slaney
Analog vs. Digital

The Analog World:
Continuous time/space, continuous valued signals or images, e.g. vinyl records, photographs, x-ray films.

The Digital World:
Discrete time/space, discrete-valued signals or images, e.g. CD-Roms, DVDs, digital photos, digital x-rays, CT, MRI, ultrasound.

Artifacts

Object

Effect of Noise

Aliasing due to insufficient number of detectors

Aliasing due to insufficient number of views

The Process of Sampling

\[ g(x) \] sample

\[ g[n]=g(n \Delta x) \]
Questions

How finely do we need to sample?

What happens if we don’t sample finely enough?

Can we reconstruct the original signal or image from its samples?
Sampling in k-space

Comb Function

\[ \text{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x - n) \]

Other names: Impulse train, bed of nails, shah function.

Scaled Comb Function

\[ \text{comb} \left( \frac{x}{\Delta x} \right) = \sum_{n=-\infty}^{\infty} \delta \left( \frac{x}{\Delta x} - n \right) \]

\[ = \sum_{n=-\infty}^{\infty} \delta \left( \frac{x - n\Delta x}{\Delta x} \right) \]

\[ = \Delta x \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x) \]

1D spatial sampling

\[ g_3(x) = g(x) \frac{1}{\Delta x} \text{comb} \left( \frac{x}{\Delta x} \right) \]

\[ = g(x) \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x) \]

\[ = \sum_{n=-\infty}^{\infty} g(n\Delta x) \delta(x - n\Delta x) \]

Recall the sifting property \( \int g(x) \delta(x - a) \, dx = g(a) \)

But we can also write \( \int g(a) \delta(x - a) = g(a) \int \delta(x - a) = g(a) \)

So, \( g(x) \delta(x - a) = g(a) \delta(x - a) \)
1D spatial sampling

\[ g(x) \]

\[ \text{comb}(x/\Delta x)/\Delta x \]

\[ g_s(x) \]

Fourier Transform of comb(x)

\[ F[\text{comb}(x)] = \text{comb}(k_x) \]

\[ = \sum_{n=-\infty}^{\infty} \delta(k_x - n) \]

\[ F\left[ \frac{1}{\Delta x} \text{comb}\left( \frac{x}{\Delta x} \right) \right] = \frac{1}{\Delta x} \text{comb}(k_x \Delta x) \]

\[ = \sum_{n=-\infty}^{\infty} \delta(k_x \Delta x - n) \]

\[ = \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} \delta(k_x - \frac{n}{\Delta x}) \]

Fourier Transform of comb(x/\Delta x)

\[ F[\text{comb}(x/\Delta x)] = \frac{1}{\Delta x} \text{comb}\left( \frac{x}{\Delta x} \right) \]

\[ = G(k_x) \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} \delta\left( k_x - \frac{n}{\Delta x} \right) \]

\[ = \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} G(k_x) \delta\left( k_x - \frac{n}{\Delta x} \right) \]

\[ = \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} G\left( k_x - \frac{n}{\Delta x} \right) \]

Fourier Transform of \( g_s(x) \)

\[ F[g_s(x)] = F\left[ g(x) \frac{1}{\Delta x} \text{comb}\left( \frac{x}{\Delta x} \right) \right] \]

\[ = G(k_x) F\left[ \frac{1}{\Delta x} \text{comb}\left( \frac{x}{\Delta x} \right) \right] \]

\[ = G(k_x) \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} \delta\left( k_x - \frac{n}{\Delta x} \right) \]

\[ = \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} G(k_x) \delta\left( k_x - \frac{n}{\Delta x} \right) \]

\[ = \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} G\left( k_x - \frac{n}{\Delta x} \right) \]
Fourier Transform of $g_S(x)$

$$G(k_x)$$

$$1/\Delta x$$

Nyquist Condition

$$G(k_x)$$

$$1/\Delta x$$

To avoid overlap, we require that $1/\Delta x > 2B$ or $K_S > 2B$ where $K_S = 1/\Delta x$ is the sampling frequency

Example

Assume that the highest spatial frequency in an object is $B = 2 \text{ cm}^{-1}$. Thus, smallest spatial period is $0.5 \text{ cm}$. Nyquist theorem says we need to sample with $\Delta x < 1/2B = 0.25 \text{ cm}$

This corresponds to 2 samples per spatial period.

Reconstruction from Samples

$$G_S(k_x)$$

$$K_S = 1/\Delta x$$

Multiply by $(1/K_S) \text{rect}(k/k_S)$

$$(1/K_S) G_S(k_x) \text{rect}(k/k_S) = G(k_x)$$
Example Cosine Reconstruction

\[
\cos(2\pi k_0 x) = \sum_{k=-\infty}^{\infty} \delta(k - k_0) \delta(k - k_0)
\]

Reconstruction from Samples

If the Nyquist condition is met, then
\[
\hat{G}_S(k) = \frac{1}{K_S} G_S(k) \text{rect}(k / K_S) = G(k)
\]

And the signal can be reconstructed by convolving the sample with a sinc function
\[
\hat{g}_S(x) = g_S(x) * \text{sinc}(K_s x)
\]

Cosine Example with \(K_S=2k_0\)

\[
\text{sinc}(K_s x) = \text{sinc}(x / \Delta x)
\]

Reconstruction from Samples

\[
g(x) \rightarrow g_S(x) \rightarrow \hat{g}_S(x)
\]
Aliasing occurs when the Nyquist condition is not satisfied. This occurs for $K_s \leq 2B$. 

\begin{align*}
G(k_s) &= k_s \\
-\B & \quad \B \\
K_s
\end{align*}
1. Consider the function \( g(x) = \cos(2\pi k_0 x) \). Sketch this function. You sample this signal in the spatial domain with a sampling rate \( K_s = 1/\Delta x \) (e.g., samples spaced at intervals of \( \Delta x \)). What is the minimum sampling rate that you can use without aliasing? Give an intuitive explanation for your answer.

Detector Sampling Requirements

- Sampling interval \( \Delta r \)
- Beamwidth \( \Delta s \)
**Smoothing of Projection**

\[ g_s(l, \theta) = \text{rect}(l/\Delta s) \ast g(l, \theta) \]

\[ G_s(k_x, \theta) = \Delta s \text{sinc}(k_x \Delta s) G(k_x, \theta) \]

**Sampling Requirements**

Detectors \( \Delta x \leq \Delta s/2 \)

Sampled Smoothed Projection

**View Aliasing**

Kak and Slaney
View Sampling Requirements

View Sampling -- how many views?

Basic idea is that to make the maximum angular sampling the same as the projection sampling.

\[
\frac{\pi \text{FOV}}{N_{\text{views}}} = \Delta r
\]

\[
N_{\text{views,360}} = \frac{\pi \text{FOV}}{\Delta r} = \pi N_{\text{proj}} \quad \text{(for 360 degrees)}
\]

\[
N_{\text{views,180}} = \frac{\pi N_{\text{proj}}}{2} \quad \text{(for 180 degrees)}
\]

Example

beamwidth \( \Delta s = 1 \) mm
Field of View (FOV) = 50 cm
\( \Delta r = \Delta s/2 = 0.5 \) mm

500 mm/ 0.5 mm = \( N = 1000 \) detector samples
\( \pi N = 3146 \) views per 360 degrees

\(~ 1500 \) views per 180 degrees

CT "Rule of Thumb"

\( N_{\text{view}} = N_{\text{detectors}} = N_{\text{pixels}} \)