Fourier Sampling

Instead of sampling the signal, we sample its Fourier Transform.

\[ G_s(k_x) = \frac{1}{\Delta k_x} \text{comb}(k_x/\Delta k_x) \]

\[ G_s(k_x) = G(k_x) \frac{1}{\Delta k_x} \text{comb}(k_x/\Delta k_x) \]

\[ = G(k_x) \sum_{n=-\infty}^{\infty} \delta(k_x - n\Delta k_x) \]

\[ = \sum_{n=-\infty}^{\infty} G(n\Delta k_x) \delta(k_x - n\Delta k_x) \]
Fourier Sampling -- Inverse Transform

\[ g_s(x) = F^{-1}[G_s(k_x)] \]
\[ = F^{-1}\left[G(k_x)\frac{1}{\Delta k_x} \text{comb}\left(\frac{k_x}{\Delta k_x}\right)\right] \]
\[ = F^{-1}[G(k_x)] + F^{-1}\left[\frac{1}{\Delta k_x} \text{comb}\left(\frac{k_x}{\Delta k_x}\right)\right] \]
\[ = g(x) + \text{comb}(x\Delta k_x) \]
\[ = g(x) + \sum_{n=-\infty}^{\infty} \delta(x - n\Delta k_x) \]
\[ = g(x) + \frac{1}{\Delta k_x} \sum_{n=-\infty}^{\infty} g(x - n\Delta k_x) \]
\[ = \frac{1}{\Delta k_x} \sum_{n=-\infty}^{\infty} g(x - n\Delta k_x) \]

Nyquist Condition

To avoid overlap, \(1/\Delta k_x > \text{FOV}\), or equivalently, \(\Delta k_x < 1/\text{FOV}\)

Aliasing

Aliasing occurs when \(1/\Delta k_x < \text{FOV}\)

Intuitive view of Aliasing

\[ k_x = 2/\text{FOV} \]
\[ k_x = 1/\text{FOV} \]
### Aliasing

\[ \Delta B(x) = G_k x \]

**Aliasing Example**

\[ \Delta k = 1 \]

\[ \frac{1}{\Delta k} = 1 \]

- Slower
- Faster

### 2D Comb Function

\[
\text{comb}(x, y) = \sum_{m} \sum_{n} \delta(x - m, y - n)
\]

\[
= \sum_{m} \sum_{n} \delta(x - m) \delta(y - n)
\]

\[
= \text{comb}(x) \text{comb}(y)
\]

### Scaled 2D Comb Function

\[
\text{comb}(x/\Delta x, y/\Delta y) = \text{comb}(x/\Delta x) \text{comb}(y/\Delta y)
\]

\[
= \Delta x \Delta y \sum_{m} \sum_{n} \delta(x - m\Delta x) \delta(y - n\Delta y)
\]

\[
\Delta x
\]

\[
\Delta y
\]
2D k-space sampling

\[ G_s(k_x, k_y) = G(k_x, k_y) \frac{1}{\Delta k_x \Delta k_y} \text{comb} \left( \frac{k_x}{\Delta k_x}, \frac{k_y}{\Delta k_y} \right) \]

\[ = G(k_x, k_y) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(k_x - m\Delta k_x, k_y - n\Delta k_y) \]

\[ = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} G(m\Delta k_x, n\Delta k_y) \delta(k_x - m\Delta k_x, k_y - n\Delta k_y) \]

Nyquist Conditions

\[
\begin{align*}
\frac{1}{\Delta k_x} & > \text{FOV}_x \\
\frac{1}{\Delta k_y} & > \text{FOV}_y \\
\text{FOV}_x & > \frac{1}{\Delta k_x} \\
\text{FOV}_y & > \frac{1}{\Delta k_y}
\end{align*}
\]
Windowing

Windowing the data in Fourier space

\[ G_w(k_x, k_y) = G(k_x, k_y)W(k_x, k_y) \]

Results in convolution of the object with the inverse transform of the window

\[ g_w(x, y) = g(x, y) * w(x, y) \]

Windowing Example

\[ W(k_x, k_y) = \text{rect}(\frac{k_x}{W_x})\text{rect}(\frac{k_y}{W_y}) \]

\[ w(x, y) = F^{-1}[\text{rect}(\frac{k_x}{W_x})\text{rect}(\frac{k_y}{W_y})] \]

\[ = W_xW_y\text{sinc}(W_x x)\text{sinc}(W_y y) \]

\[ g_w(x, y) = g(x, y) * W_xW_y\text{sinc}(W_x x)\text{sinc}(W_y y) \]
Windowing Example

\[ g(x, y) = (\delta(x) + \delta(x - 1))h(y) \]
\[ g_w(x, y) = (\delta(x) + \delta(x - 1))h(y) * W_kx W_ky \sin(W_kx x) \sin(W_ky y) \]
\[ = W_kx W_ky \left( \delta(x) + \delta(x - 1) \right) \sin(W_kx x) \sin(W_ky y) \]
\[ = W_kx W_ky \left( \sin(W_kx x) + \sin(W_kx (x - 1)) \right) \sin(W_ky y) \]

Resolution and spatial frequency

With a window of width \( W_kx \), the highest spatial frequency is \( W_kx / 2 \).
This corresponds to a spatial period of \( 2 / W_kx \).

\[ \frac{1}{W_kx} = \text{Effective Width} \]

Sampling and Windowing

Effective Width

\[ w_E = \frac{1}{w(0)} \int_{-\infty}^{\infty} w(x) dx \]

Example

\[ w_w = \frac{1}{1} \int_{-\infty}^{\infty} \text{sinc}(W_kx x) dx \]
\[ = \text{sinc}(W_kx x) \left[ \frac{1}{W_kx} \right]_{-\infty}^{\infty} \]
\[ = \frac{1}{W_kx} \]

\[ \frac{1}{W_kx} \]

\[ \frac{1}{W_kx} \]

\[ \frac{1}{W_kx} \]

\[ \frac{1}{W_kx} \]
Sampling and Windowing

Sampling and windowing the data in Fourier space

\[ G_{xy}(k_x, k_y) = G(k_x, k_y) \frac{1}{\Delta k_x \Delta k_y} \text{comb} \left( \frac{k_x}{\Delta k_x}, \frac{k_y}{\Delta k_y} \right) \text{rect} \left( \frac{k_x}{W_{k_x}}, \frac{k_y}{W_{k_y}} \right) \]

Results in replication and convolution in object space.

\[ g_{xy}(x, y) = W_{k_x} W_{k_y} g(x, y) * \text{comb}(\Delta k_x, \Delta k_y) * \text{sinc}(W_{k_x} x) \text{sinc}(W_{k_y} y) \]

Sampling in \( k_x \)

\[ k_x = \frac{M}{2\Delta k_x} \]

\[ \text{FOV}_x = \frac{1}{\Delta k_x} \]

Sampling in \( k_y \)

\[ k_y = \frac{N}{2\Delta k_y} \]

\[ \text{FOV}_y = \frac{1}{\Delta k_y} \]

Sampling in \( k_x \)

One I,Q sample every \( \Delta t \)

\[ M = I + jQ \]

Note: In practice, there are number of ways of implementing this processing.

Sampling in \( k_y \)

\[ \Delta k_y = \frac{\gamma}{2\pi} G_{xy} \tau_y \]
Goal:

\[ \text{FOV}_x = \text{FOV}_y = 25.6 \text{ cm} \]

\[ \delta_x = \delta_y = 0.1 \text{ cm} \]

Readout Gradient:

\[ \frac{1}{W_x} = \frac{1}{2k_{x,max}} = \frac{1}{\frac{\gamma}{2\pi} G_{x} \tau_x} \]

\[ \tau_x = \frac{1}{\delta_x} \frac{\gamma}{2\pi} G_{x} = \frac{1}{(0.1 \text{ cm})(4257 \text{ G/cm}^2)(0.28675 \text{ G/cm})} = 8.192 \text{ ms} = N_{\text{read}} \Delta t \]

where

\[ N_{\text{read}} = \frac{\text{FOV}}{\delta_x} = 256 \]

Phase-Encode Gradient:

\[ \frac{1}{W_y} = \frac{1}{2k_{y,max}} = \frac{1}{\frac{\gamma}{2\pi} G_{y} \tau_y} \]

\[ \tau_y = \frac{1}{\delta_y} \frac{\gamma}{2\pi} G_{y} = \frac{1}{(0.1 \text{ cm})(4257 \text{ G/cm}^2)(0.28675 \text{ G/cm})} = 2.402 \times 10^{-3} \text{ s} \]

\[ = 2.2402 \times 10^{-7} \text{ T/cm} \]

\[ = 0.00224 \text{ G/cm} \]
**Example**

Phase-Encode Gradient:

\[
\delta_y = \frac{1}{2\pi} 2 G_y \tau_y
\]

\[
G_y = \frac{1}{\delta_y} \frac{2 \tau_y}{2\pi} = \frac{1}{0.1 \text{cm}(4257 \text{ G} \cdot \text{s}^2 / \text{cm}^2)(4.096 \times 10^{-3} \text{s})} = 0.2868 \text{ G/cm}
\]

\[
N_p = \frac{\text{FOV}}{\delta_y} = 256
\]

**Sampling**

In practice, an even number (typically power of 2) sample is usually taken in each direction to take advantage of the Fast Fourier Transform (FFT) for reconstruction.

**Example**

Consider the k-space trajectory shown below. ADC samples are acquired at the points shown with \( t = 10 \mu\text{sec} \). The desired FOV (both x and y) is 10 cm and the desired resolution (both x and y) is 2.5 cm. Draw the gradient waveforms required to achieve the k-space trajectory. Label the waveform with the gradient amplitudes required to achieve the desired FOV and resolution. Also, make sure to label the time axis correctly.

![Diagram of k-space trajectory and gradient waveforms](image)
**Gibbs Artifact**

256x256 image

256x128 image

Images from http://www.mritutor.org/mritutor/gibbs.htm

**Apodization**

\[ h(k_x) = \frac{1}{2}(1 + \cos(2\pi k_x)) \]

\[ = 0.5\text{sinc}(x) + 0.25\text{sinc}(x-1) + 0.25\text{sinc}(x+1) \]

**Aliasing and Bandwidth**

Temporal filtering in the readout direction limits the readout FOV. So there should never bealiasing in the readout direction.

**Aliasing and Bandwidth**

Lowpass filter in the readout direction to prevent aliasing.