**Bioengineering 280A**  
Principles of Biomedical Imaging  
Fall Quarter 2008  
CT/Fourier Lecture 3  

**Topics**

- Modulation  
- Modulation Transfer Function  
- Convolution/Multiplication  
- Revisit Projection-Slice Theorem  
- Filtered Backprojection

**Modulation**

\[
F\left[ g(x)e^{j2\pi f_0 x} \right] = G(k_x) * \delta(k_x - k_0) = G(k_x - k_0)
\]

\[
F\left[ g(x)\cos(2\pi k_0 x) \right] = \frac{1}{2} G(k_x - k_0) + \frac{1}{2} G(k_x + k_0)
\]

\[
F\left[ g(x)\sin(2\pi k_0 x) \right] = \frac{1}{2j} G(k_x - k_0) - \frac{1}{2j} G(k_x + k_0)
\]

**Example**

Amplitude Modulation (e.g. AM Radio)

\[
g(t) \quad \rightarrow \quad 2g(t) \cos(2\pi f_0 t)
\]

\[
2\cos(2\pi f_0 t)
\]

\[
G(f) \quad \rightarrow \quad G(f-f_0) + G(f+f_0)
\]
Modulation Example

Modulation Transfer Function (MTF) or Frequency Response

Modulation Transfer Function
Eigenfunctions

The fundamental nature of the convolution theorem may be better understood by observing that the complex exponentials are eigenfunctions of the convolution operator.

\[ e^{j2\pi k_x x} \]

\[ z(x) = g(x) * e^{j2\pi k_x x} = \int_{-\infty}^{\infty} g(u)e^{j2\pi k_x (x-u)}du = G(k_x)e^{j2\pi k_x x} \]

The response of a linear shift invariant system to a complex exponential is simply the exponential multiplied by the FT of the system’s impulse response.

MTF = Fourier Transform of PSF

Convolution/Multiplication

Now consider an arbitrary input \( h(x) \).

\[ h(x) \]

Recall that we can express \( h(x) \) as the integral of weighted complex exponentials.

\[ h(x) = \int_{-\infty}^{\infty} H(k_x)e^{j2\pi k_x x}dk_x \]

Each of these exponentials is weighted by \( G(k_x) \) so that the response may be written as

\[ z(x) = \int_{-\infty}^{\infty} G(k_x)H(k_x)e^{j2\pi k_x x}dk_x \]

Convolution/Modulation Theorem

\[ F\{ g(x) * h(x) \} = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} g(u)h(x-u)du \right] e^{-j2\pi k_x x}dx = \int_{-\infty}^{\infty} g(u) \int_{-\infty}^{\infty} h(x-u)e^{-j2\pi k_x x}dxdu = \int_{-\infty}^{\infty} g(u)H(k_x)e^{-j2\pi k_x u}du = G(k_x)H(k_x) \]

Convolution in the spatial domain transforms into multiplication in the frequency domain. Dual is modulation

\[ F\{ g(x)h(x) \} = G(k_x) * H(k_x) \]
2D Convolution/Multiplication

Convolution
\[ F[g(x,y) * h(x,y)] = G(k_x,k_y)H(k_x,k_y) \]

Multiplication
\[ F[g(x,y)b(x,y)] = G(k_x,k_y)\ast H(k_x,k_y) \]

Application of Convolution Thm.
\[ \Lambda(x) = \begin{cases} 1-|x| & 0 \leq |x| < 1 \\ 0 & \text{otherwise} \end{cases} \]
\[ F(\Lambda(x)) = \int_{-\infty}^{\infty} (1-|x|)e^{-j\omega x}dx = ?? \]

Convolution Example
Response of an Imaging System

\[ z(x,y) = g(x,y) \ast h_1(x,y) \ast h_2(x,y) \ast h_3(x,y) \]

\[ Z(k_x, k_y) = G(k_x, k_y) \cdot H_1(k_x, k_y) \cdot H_2(k_x, k_y) \cdot H_3(k_x, k_y) \]

System MTF = Product of MTFs of Components

Useful Approximation

\[ FWHM_{\text{System}} = \sqrt{FWHM_1^2 + FWHM_2^2 + \cdots + FWHM_N^2} \]

Example

\[ FWHM_1 = 1 \text{ mm} \]
\[ FWHM_2 = 2 \text{ mm} \]
\[ FWHM_{\text{System}} = \sqrt{5} = 2.24 \text{ mm} \]

8. Referring to Figure 1 (above) which demonstrates 3 different line spread functions (LSFs), which LSF will yield the best spatial resolution?

10. Referring to Figure 1 which shows LSFs, and Figure 2 which shows the corresponding modulation transfer functions (MTFs), which MTF corresponds to LSF C?

A. MTF number 1
B. MTF number 2
C. MTF number 3

D74. The intrinsic resolution of a gamma camera is 5 mm. The collimator resolution is 10 mm. The overall system resolution is _____ mm.

A. 15
B. 11.2
C. 7.5
D. 5.0
E. 0.5
Projection Theorem

\[ U(k_x, 0) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mu(x, y) e^{-2\pi i (k_x x + k_y y)} \, dx \, dy \]

\[ = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} \mu(x, y) \, dy \right] e^{-2\pi i k_x x} \, dx \]

\[ = \int_{-\infty}^{\infty} g(0) e^{-2\pi i k_x x} \, dx \]

\[ = \int_{-\infty}^{\infty} g(0) e^{-2\pi i \rho \sin \theta} \, d\rho \]

In-Class Example:
\[ \mu(x, y) = \cos(2\pi x) \]

Projection Slice Theorem

\[ G(\rho, \theta) = \int_{-\infty}^{\infty} g(l, \theta) e^{-2\pi i \rho \sin \theta} \, dl \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \delta(x \cos \theta + y \sin \theta - l) e^{-2\pi i \rho \sin \theta} \, dx \, dy \, dl \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) e^{-2\pi i \rho (x \cos \theta + y \sin \theta)} \, dx \, dy \]

\[ = F_{2D} \left[ f(x, y) \right] \delta_{\rho = \rho \sin \theta} \]

Fourier Reconstruction

Interpolate onto Cartesian grid then take inverse transform
Polar Version of Inverse FT

\[ \mu(x, y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(k_x, k_y) e^{j2\pi(k_x x + k_y y)} dk_x dk_y \]

\[ = \int_{0}^{2\pi} \int_{0}^{\infty} G(k, \theta)e^{j2\pi(k \cdot (x \cos \theta + y \sin \theta)} |k| dk d\theta \]

Note:

\[ g(1, \theta + \pi) = g(-1, \theta) \]

So

\[ G(k, \theta + \pi) = G(-k, \theta) \]

Filtered Backprojection

\[ \mu(x, y) = \int_{0}^{\pi} \int_{0}^{\infty} G(k, \theta)e^{j2\pi(k \cdot (x \cos \theta + y \sin \theta)} |k| dk d\theta \]

\[ = \int_{0}^{\pi} \int_{0}^{\infty} |k| G(k, \theta)e^{j2\pi m \cdot (x \cos \theta + y \sin \theta)} k dk d\theta \]

where \( l = x \cos \theta + y \sin \theta \)

\[ g'(l, \theta) = \int_{0}^{\infty} |k| G(k, \theta)e^{j2\pi m \cdot k} dk \]

\[ = g(l, \theta) \ast F^{-1}[|k|] \]

\[ = g(l, \theta) \ast q(l) \]

Fourier Interpretation

Density = \( \frac{N}{\text{circumference}} = \frac{N}{2\pi |k|} \)

Low frequencies are oversampled. So to compensate for this, multiply the k-space data by |k| before inverse transforming.

Ram-Lak Filter

\( k_{max} = 1/\Delta s \)
Reconstruction Path

Projection

\[ F \]

\[ F^{-1} \]

Filtered Projection
Back-Project

Example

Figure 6.15
Convolution step:
(a) Original sinogram;
(b) filtered sinogram;
(c) profile of sinogram row [white line in (a)]; and
(d) profile of filtered sinogram row [white line in (b)].
Example