HOMEWORK #4
Due at the start of Class on Thursday 10/30/08

Readings:
Section 2.8 and review Chapter 6 as necessary.

Problems:
1. Generalized functions. Recall that delta functions are not ordinary functions, and are defined by what they “do”. In class, we showed how to integrate a delta function with a “test” function in order to see what it does. Using this approach, show that
\[
\delta(x) = \int_{-\infty}^{\infty} e^{j2\pi k x} dx .
\]
HINT: Multiply the expression by a test function \( g(x) \) and integrate over \( x \); then consider that the resulting expression is in the form of an inverse Fourier transform evaluated at a specific \( x \) location. This will allow you to show that \( \int_{-\infty}^{\infty} e^{j2\pi k x} dx \) “acts like \( \delta(x) \).

2. Let \( G(k, \theta) \) be the 1-D Fourier transform of the projection \( g(l, \theta) \).
   a) Show that \( g(l, \theta + \pi) = g(-l, \theta) \)
   b) Next, show that \( G(k, \theta + \pi) = G(-k, \theta) \)

3. Problem 2.24

4. Consider the CT k-space filter \( G(k) = \|w(k) \) where \( w(k) \) is a windowing function. For each of the following window functions, use MATLAB to plot the k-space filter and then derive its inverse Fourier transform.
   a) The Ram-Lak Filter with \( w(k) = \text{rect} \left( \frac{k}{2k_{\max}} \right) \).
   b) A Hanning window defined as \( w(k) = \text{rect} \left( \frac{k}{2k_{\max}} \right) 0.5 + 0.5 \cos \left( \frac{\pi k}{k_{\max}} \right) \).
   c) Use MATLAB to plot out and compare the inverse transforms from parts (a) and (b). Comment on the relative advantages and disadvantages of the two filters for CT reconstruction.

5. A parallel beam CT imaging system is used to image an object defined as:
\[
f(x, y) = \text{rect}(x, y) + \left( \text{rect}(x, y) \ast \left[ \delta(x - 3) + \delta(x + 3) \right] \ast \left[ \delta(y - 4) + \delta(y + 4) \right] \right)
\]
   a) Sketch the object and draw the projections of the object at 0 degrees and 45 degrees.
   b) Derive the Fourier transform of the object
   c) Show that the Projection-slice theorem holds for the projections at 0 and 45 degrees.

6. (20 pts) Consider the object \( f(x, y) = \cos \left( \frac{2}{\sqrt{3}} \pi x + 2\pi y \right) \)
   a) Sketch the object.
   b) Consider sampling the object in both the x and y directions with sample intervals of \( \Delta_x \) and \( \Delta_y \), respectively. Indicate what sample intervals should be used to avoid aliasing.
   c) Now consider imaging the object with a parallel beam CT imaging system. At what angle will the projection be non-zero?
   d) We now wish to sample the non-zero projection. What sampling interval should we use to avoid aliasing?
e) Now consider the object \( g(x,y) = (f(x,y))^2 \). Answer items (c) and (d) for this object.