Linearity (Addition)

A system $R$ is linear if for two inputs $I_1(x,y)$ and $I_2(x,y)$ with outputs $R(I_1(x,y))=K_1(x,y)$ and $R(I_2(x,y))=K_2(x,y)$

the response to the weighted sum of inputs is the weighted sum of outputs:

$R(a_1I_1(x,y)+a_2I_2(x,y))=a_1K_1(x,y)+a_2K_2(x,y)$

Linearity (Scaling)

Linearity

A system $R$ is linear if for two inputs $I_1(x,y)$ and $I_2(x,y)$ with outputs $R(I_1(x,y))=K_1(x,y)$ and $R(I_2(x,y))=K_2(x,y)$

the response to the weighted sum of inputs is the weighted sum of outputs:

$R(a_1I_1(x,y)+a_2I_2(x,y))=a_1K_1(x,y)+a_2K_2(x,y)$
Example
Are these linear systems?
\[ g(x,y) + 10 \]
\[ 10g(x,y) \]
\[ h(x-1,y-1) \]

Superposition
\[ g[m] = g[0]\delta[m] + g[1]\delta[m-1] + g[2]\delta[m-2] \]
\[ h[m',k] = L[\delta[m-k]] \]
\[ y[m'] = L[g[m]] \]
\[ = L[g[0]\delta[m] + g[1]\delta[m-1] + g[2]\delta[m-2]] \]
\[ = L[g[0]\delta[m]] + L[g[1]\delta[m-1]] + L[g[2]\delta[m-2]] \]
\[ = g[0]L[\delta[m]] + g[1]L[\delta[m-1]] + g[2]L[\delta[m-2]] \]
\[ = g[0]h[m',0] + g[1]h[m',1] + g[2]h[m',2] \]
\[ = \sum_{k=0}^{2} g[k]h[m',k] \]

Superposition Integral
What is the response to an arbitrary function \( g(x_1,y_1) \)?
Write \( g(x_1,y_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi,\eta)\delta(x_1-\xi,y_1-\eta)d\xi d\eta. \)
The response is given by
\[ h(x_2,y_2) = L[g(x_1,y_1)] \]
\[ = L[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi,\eta)\delta(x_1-\xi,y_1-\eta)d\xi d\eta] \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi,\eta)L[\delta(x_1-\xi,y_1-\eta)]d\xi d\eta \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi,\eta)h(x_2,y_2;\xi,\eta)d\xi d\eta \]

Space Invariance
If a system is space invariant, the impulse response depends only on the difference between the output coordinates and the position of the impulse and is given by
\[ h(x_2,y_2;\xi,\eta) = h(x_2-\xi,y_2-\eta) \]
**Pinhole Magnification Example**

In this example, the pinhole system is space invariant.

![Pinhole Diagram](image)

**Convolution**

\[ g[m] = g[0] \delta[m] + g[1] \delta[m - 1] + g[2] \delta[m - 2] \]

\[ h[m', k] = L[ \delta[m - k] ] = h[m' - k] \]

\[ y[m'] = \sum_{k=0}^{2} g[k] h[m' - k] \]

**1D Convolution**

\[ I(x) = \int_{-\infty}^{\infty} g(\xi) h(x - \xi) d\xi \]

\[ = \int_{-\infty}^{\infty} g(\xi) h(x - \xi) d\xi \]

\[ = g(x) * h(x) \]

Useful fact:

\[ g(x) * \delta(x - \Delta) = \int_{-\infty}^{\infty} g(\xi) \delta(x - \Delta - \xi) d\xi \]

\[ = g(x - \Delta) \]

**2D Convolution**

For a space invariant linear system, the superposition integral becomes a convolution integral.

\[ I(x_2, y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi, \eta) h(x_2 - \xi, y_2 - \eta) d\xi d\eta \]

\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi, \eta) h(x_2 - \xi, y_2 - \eta) d\xi d\eta \]

\[ = g(x_2, y_2) ** h(x_2, y_2) \]

where ** denotes 2D convolution. This will sometimes be abbreviated as *, e.g. \( I(x_2, y_2) = g(x_2, y_2) * h(x_2, y_2) \).
**Rectangle Function**

\[ \Pi(x) = \begin{cases} 
0 & |x| > \frac{1}{2} \\
1 & |x| \leq \frac{1}{2}
\end{cases} \]

Also called \( \text{rect}(x) \)

\[ \Pi(x,y) = \Pi(x)\Pi(y) \]

**1D Convolution Examples**

\[ \begin{array}{c}
\hline \\
& \frac{1}{2} & \\
-\frac{1}{2} & 1 & \frac{1}{2} \\
\hline \\
\end{array} \]

\[ x \ast \begin{array}{c}
\hline \\
& \frac{1}{2} & \\
-\frac{1}{2} & 1 & \frac{1}{2} \\
\hline \\
\end{array} \]

**2D Convolution Example**

\[ g(x) = \delta(x+\frac{1}{2},y) + \delta(x,y) \]
\[ h(x) = \text{rect}(x,y) \]

\[ I(x,y) = g(x) \ast h(x,y) \]

**2D Convolution Example**
For off-center pinhole object, the shifted source image can be written as

\[ s\left(\frac{x-Mx_0}{m}\right) = s\left(\frac{x}{m}\right) \frac{1}{M} \delta\left(\frac{x-Mx_0}{M}\right) \]

\[ = s(x/m) \ast \left(\frac{x}{M}\right) \]

For the general 2D case, we convolve the magnified object with the impulse response

\[ I(x,y) = t\left(\frac{x}{M}, \frac{y}{M}\right) \ast \ast \frac{1}{m^2} \ast s\left(\frac{x}{m}, \frac{y}{m}\right) \]

Note: we have ignored obliquity factors etc.
X-Ray Imaging

$m = 1, M = 2$

\[
\frac{1}{m} \text{rect} \left( \frac{x}{m} \right) \ast \left( \frac{x}{M} \right) = \text{rect}(x/10) \ast \text{rect}(x/20)
\]

= ???

Summary

1. The response to a linear system can be characterized by a spatially varying impulse response and the application of the superposition integral.
2. A shift invariant linear system can be characterized by its impulse response and the application of a convolution integral.