Relaxation

An excitation pulse rotates the magnetization vector away from its equilibrium state (purely longitudinal). The resulting vector has both longitudinal $M_z$ and transverse $M_{xy}$ components.

Due to thermal interactions, the magnetization will return to its equilibrium state with characteristic time constants.

- $T_1$ spin-lattice time constant, return to equilibrium of $M_z$
- $T_2$ spin-spin time constant, return to equilibrium of $M_{xy}$

Longitudinal Relaxation

$$\frac{dM_z}{dt} = -\frac{M_z - M_0}{T_1}$$

After a 90 degree pulse, $M_z(t) = M_0(1 - e^{-t/T_1})$

Due to exchange of energy between nuclei and the lattice (thermal vibrations). Process continues until thermal equilibrium as determined by Boltzmann statistics is obtained.

The energy $\Delta E$ required for transitions between down to up spins, increases with field strength, so that $T_1$ increases with $B$.

T1 Values

Image, caption: Nishimura, Fig. 4.2
Transverse Relaxation

\[ \frac{dM_{xy}}{dt} = -\frac{M_{xy}}{T_2} \]

Each spin’s local field is affected by the z-component of the field due to other spins. Thus, the Larmor frequency of each spin will be slightly different. This leads to a dephasing of the transverse magnetization, which is characterized by an exponential decay.

\[ T_2 \] is largely independent of field. \( T_2 \) is short for low frequency fluctuations, such as those associated with slowly tumbling macromolecules.

T2 Relaxation

After a 90 degree excitation

\[ M_{xy}(t) = M_0 e^{-t/T_2} \]

T2 Values

<table>
<thead>
<tr>
<th>Tissue</th>
<th>( T_2 ) (ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>gray matter</td>
<td>100</td>
</tr>
<tr>
<td>white matter</td>
<td>92</td>
</tr>
<tr>
<td>muscle</td>
<td>47</td>
</tr>
<tr>
<td>fat</td>
<td>85</td>
</tr>
<tr>
<td>kidney</td>
<td>58</td>
</tr>
<tr>
<td>liver</td>
<td>43</td>
</tr>
<tr>
<td>CSF</td>
<td>4000</td>
</tr>
</tbody>
</table>

Solids exhibit very short \( T_2 \) relaxation times because there are many low frequency interactions between the immobile spins. On the other hand, liquids show relatively long \( T_2 \) values, because the spins are highly mobile and net fields average out.
Questions: How can one achieve T2 weighting? What are the relative T2’s of the various tissues?

Bloch Equation

\[
\frac{d\mathbf{M}}{dt} = \mathbf{M} \times \gamma \mathbf{B} - \frac{M_x i + M_y j}{T_2} - \frac{(M_z - M_0) k}{T_1}
\]

\( i, j, k \) are unit vectors in the x,y,z directions.

Free precession about static field

\[
\frac{d\mathbf{M}}{dt} = \mathbf{M} \times \gamma \mathbf{B} = \gamma M_x i + M_y j + M_z k = \gamma \frac{\hat{i} (B_x M_x - B_y M_y) + \hat{j} (B_x M_x - B_z M_z) + \hat{k} (B_y M_y - B_z M_z)}{B_x B_y B_z}
\]
Free precession about static field

\[
\begin{bmatrix}
\frac{dM_x}{dt} & \frac{dM_y}{dt} & \frac{dM_z}{dt}
\end{bmatrix}
= \begin{bmatrix}
B_x M_y - B_y M_x & B_y M_z - B_z M_y & B_z M_x - B_x M_z \\
B_x & B_y & B_z \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
\]

\[
\frac{dM_x}{dt} = \gamma (B_y M_z - B_z M_y) \\
\frac{dM_y}{dt} = \gamma (B_z M_x - B_x M_z) \\
\frac{dM_z}{dt} = \gamma (B_x M_y - B_y M_x)
\]

Matrix Form with $B=\mathbf{B}_0$

\[
\begin{bmatrix}
\frac{dM_x}{dt} & \frac{dM_y}{dt} & \frac{dM_z}{dt}
\end{bmatrix}
= \begin{bmatrix}
-\gamma B_0 & 0 & 0 \\
0 & -\gamma B_0 & 0 \\
B_x & B_y & B_z
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
\]

Precession

\[
\begin{bmatrix}
\frac{dM_x}{dt} & \frac{dM_y}{dt} & \frac{dM_z}{dt}
\end{bmatrix}
= \begin{bmatrix}
0 & B_0 & 0 \\
-\gamma B_0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
\]

Useful to define $M = M_x + jM_y$

\[
\frac{dM}{dt} = d/\text{d}t(M_x + iM_y)
\]

Solution is a time-varying phasor

\[M(t) = M(0)e^{-j\omega_0 t} = M(0)e^{-j\omega_0 t}\]

Z-component solution

\[M_z(t) = M_0 + (M_z(0) - M_0)e^{-t/T_1}\]

Saturation Recovery

If $M_z(0) = 0$ then $M_z(t) = M_0(1 - e^{-t/T_1})$

Inversion Recovery

If $M_z(0) = -M_0$ then $M_z(t) = M_0(1 - 2e^{-t/T_1})$

Question: which way does this rotate with time?
Summary

1) Longitudinal component recovers exponentially.
2) Transverse component precesses and decays exponentially.

Summary

1) Longitudinal component recovers exponentially.
2) Transverse component precesses and decays exponentially.

Fact: Can show that $T_2 < T_1$ in order for $|M(t)| \leq M_0$.

Physically, the mechanisms that give rise to $T_1$ relaxation also contribute to transverse $T_2$ relaxation.

Gradients

Spins precess at the Larmor frequency, which is proportional to the local magnetic field. In a constant magnetic field $B_z = B_0$, all the spins precess at the same frequency (ignoring chemical shift).

Gradient coils are used to add a spatial variation to $B_z$ such that $B_z(x,y,z) = B_0 + \Delta B_z(x,y,z)$. Thus, spins at different physical locations will precess at different frequencies.
Imaging: localizing the NMR signal

The local precession frequency can be changed in a position-dependent way by applying linear field gradients.

\[ \Delta B(x) \]

Credit: R. Buxton

Gradient Fields

\[ B_z(x, y, z) = B_0 + \frac{\partial B_z}{\partial x} x + \frac{\partial B_z}{\partial y} y + \frac{\partial B_z}{\partial z} z \]

\[ = B_0 + G_x x + G_y y + G_z z \]

\[ G_z = \frac{\partial B_z}{\partial z} > 0 \]
\[ G_y = \frac{\partial B_z}{\partial y} > 0 \]

Interpretation

\[ \Delta B_x(x) = G_x x \]

Spins Precess at \( \gamma B_0 \) (slower)

Spins Precess at \( \gamma B_0 + \gamma \Delta B(x) \) (faster)


\[ \nu(x) = \gamma B_0 + \gamma \Delta B(x) \]
Rotating Frame of Reference

Reference everything to the magnetic field at isocenter.

Spins

There is nothing that nuclear spins will not do for you, as long as you treat them as human beings.
Erwin Hahn

Phasors

Phasor Diagram

\[ G(k_x) = \int_{-\infty}^{\infty} g(x) \exp(-j2\pi k_x x) dx \]

\[ \theta = -2\pi k_x x \]

\[ \theta = 0, \quad x = 0 \]
\[ 2\pi k_x x = 0 \]
\[ 2\pi k_x x = \pi / 2 \]
\[ 2\pi k_x x = \pi \]
\[ 2\pi k_x x = 3\pi / 4 \]

\[ \theta = \pi / 2, \quad x = 1 / 2 \]
\[ 2\pi k_x x = \pi / 2 \]
\[ 2\pi k_x x = \pi \]
\[ 2\pi k_x x = 3\pi / 4 \]
Interpretation

\[
\begin{align*}
\exp\left(-j2\pi \left( \frac{0}{\Delta x} \right) \right) & \\
\exp\left(-j2\pi \left( \frac{1}{\Delta x} \right) \right) & \\
\exp\left(-j2\pi \left( \frac{2}{\Delta x} \right) \right) & \\
\end{align*}
\]

Slower \quad \Delta B(x)=G_{x}x \quad Faster

Fig 3.12 from Nishimura

Hanson 2009
Phase with time-varying gradient

K-space trajectory

Hanson 2009
K-space trajectory

$G_x(t)$

$G_y(t)$

$t_1$

$t_2$

$t$

$k_x$

$k_y$

Spin-Warp

$G_x(t)$

$G_y(t)$

$k_x$

$k_y$

k-space

Image space

k-space

y

x

Fourier Transform

TT Liu, BE280A, UCSD Fall 2010
Spin-Warp Pulse Sequence

\[ G_x(t) \]

\[ G_y(t) \]

\[ G_z(t) \]

RF

Gradient Fields

Define

\[ \vec{G} = G_x \hat{i} + G_y \hat{j} + G_z \hat{k} \]

\[ \vec{r} = x \hat{i} + y \hat{j} + z \hat{k} \]

So that

\[ G_x x + G_y y + G_z z = \vec{G} \cdot \vec{r} \]

Also, let the gradient fields be a function of time. Then the z-directed magnetic field at each point in the volume is given by:

\[ B_z(\vec{r}, t) = B_0 + \vec{G}(t) \cdot \vec{r} \]
Static Gradient Fields

In a uniform magnetic field, the transverse magnetization is given by:

\[ M(t) = M(0) e^{-j\omega_0 t} e^{-j\theta(t)} \]

In the presence of non time-varying gradients we have

\[ M(\vec{r}) = M(\vec{r}, 0) e^{-j\theta(\vec{r}, t)} e^{-j\gamma t/2} \]

Time-Varying Gradient Fields

In the presence of time-varying gradients the frequency as a function of space and time is:

\[ \omega(\vec{r}, t) = \gamma B_0(\vec{r}, t) + \gamma \vec{G}(t) \cdot \vec{r} \]

Phase

Phase = angle of the magnetization phasor
Frequency = rate of change of angle (e.g. radians/sec)
Phase = time integral of frequency

\[ \psi(\vec{r}, t) = -\int_0^t \omega(\vec{r}, \tau) d\tau \]

\[ = -\omega_0 t + \Delta \psi(\vec{r}, t) \]

Where the incremental phase due to the gradients is

\[ \Delta \psi(\vec{r}, t) = \int_0^t \Delta \omega(\vec{r}, \tau) d\tau \]

\[ = -\int_0^t \vec{G}(\vec{r}, \tau) \cdot \vec{r} d\tau \]

Phase with constant gradient

If \( \Delta \omega \) is non-time varying,

\[ \Delta \psi(\vec{r}, t) = \int_0^t \Delta \omega(\vec{r}, \tau) d\tau \]

\[ = -\Delta \omega(\vec{r}) \frac{t^2}{2} \]
Time-Varying Gradient Fields

The transverse magnetization is then given by

\[ M(\tilde{r}, t) = M(\tilde{r}, 0)e^{-i T_2 \tilde{r} \cdot \tilde{G}(\tilde{r})}e^{i \omega_0 t} \]

\[ = M(\tilde{r}, 0)e^{-i T_2 \tilde{r} \cdot \tilde{G}(\tilde{r})}e^{i \omega_0 t} \exp \left( -j \int_0^t \Delta \omega(\tilde{r}, t) d\tau \right) \]

\[ = M(\tilde{r}, 0)e^{-i T_2 \tilde{r} \cdot \tilde{G}(\tilde{r})}e^{i \omega_0 t} \exp \left( -j \gamma \int_0^t \tilde{G}(\tau) \cdot \tilde{r} d\tau \right) \]

Signal Equation

Demodulate the signal to obtain

\[ s(t) = e^{i \omega_0 t} s_0(t) \]

\[ = \int \int_{x,y} m(x,y) \exp \left( -j \int_0^t \tilde{G}(\tau) \cdot \tilde{r} d\tau \right) dxdy \]

\[ = \int \int_{x,y} m(x,y) \exp \left( -j \gamma \int_0^t \tilde{G}(\tau) \cdot \tilde{r} d\tau \right) dxdy \]

\[ = \int \int_{x,y} m(x,y) \exp \left( -j 2\pi \left( k_x(t) x + k_y(t) y \right) \right) dxdy \]

\[ = \int \int_{x,y} m(x,y) \exp \left( -j 2\pi \left( k_x(t) x + k_y(t) y \right) \right) dxdy \]

\[ = M(k_x(t), k_y(t)) \]

\[ = F[m(x,y)]_{k_x(t), k_y(t)} \]

MR signal is Fourier Transform

Signal from a volume

\[ s_0(t) = \int \int_{x,y} m(\tilde{r}, t) dxdy \]

\[ = \int \int_{x,y,z} m(x,y,z,0) e^{-i T_2 \tilde{r} \cdot \tilde{G}(\tilde{r})} e^{i \omega_0 t} \exp \left( -j \gamma \int_0^t \tilde{G}(\tau) \cdot \tilde{r} d\tau \right) dxdydz \]

For now, consider signal from a slice along z and drop the T_2 term. Define

\[ m(x,y) = \int \int_{x,y} m(x,y,z,0) e^{-i T_2 \tilde{r} \cdot \tilde{G}(\tilde{r})} e^{i \omega_0 t} \exp \left( -j \gamma \int_0^t \tilde{G}(\tau) \cdot \tilde{r} d\tau \right) dxdy \]

To obtain

\[ s_0(t) = \int \int_{x,y} m(x,y) e^{-i \omega_0 t} \exp \left( -j \gamma \int_0^t \tilde{G}(\tau) \cdot \tilde{r} d\tau \right) dxdy \]

\[ = \int \int_{x,y} m(x,y) e^{-i \omega_0 t} \exp \left( -j \gamma \int_0^t \tilde{G}(\tau) \cdot \tilde{r} d\tau \right) dxdy \]

\[ = \int \int_{x,y} m(x,y) \exp \left( -j 2\pi \left( k_x(t) x + k_y(t) y \right) \right) dxdy \]

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\[ = M(k_x(t), k_y(t)) \]

\[ = F[m(x,y)]_{k_x(t), k_y(t)} \]
Recap

- Frequency = rate of change of phase.
- Higher magnetic field -> higher Larmor frequency -> phase changes more rapidly with time.
- With a constant gradient $G_x$, spins at different x locations precess at different frequencies -> spins at greater x-values change phase more rapidly.
- With a constant gradient, distribution of phases across x locations changes with time. (phase modulation)
- More rapid change of phase with x -> higher spatial frequency $k_x$.

K-space

At each point in time, the received signal is the Fourier transform of the object

$$s(t) = M[k_x(t), k_y(t)] = F[m(x,y)]_{k_x(t), k_y(t)}$$

evaluated at the spatial frequencies:

$$k_x(t) = \frac{\gamma}{2\pi} \int_0^t G_x(\tau)d\tau$$
$$k_y(t) = \frac{\gamma}{2\pi} \int_0^t G_y(\tau)d\tau$$

Thus, the gradients control our position in k-space. The design of an MRI pulse sequence requires us to efficiently cover enough of k-space to form our image.

Units

Spatial frequencies ($k_x$, $k_y$) have units of 1/distance. Most commonly, 1/cm

Gradient strengths have units of (magnetic field)/distance. Most commonly G/cm or mT/m

$\gamma/(2\pi)$ has units of Hz/G or Hz/Tesla.

$$k_x(t) = \frac{\gamma}{2\pi} \int_0^t G_x(\tau)d\tau$$
$$= [Hz/Gauss][Gauss/cm][sec]$$
$$= [1/cm]$$
**Example**

\[ G_x(t) = 1 \text{ Gauss/cm} \]

\[
k_y(t_2) = \frac{\pi}{2 \lambda} \int G_x(\tau) \, d\tau
= 4257 \text{Hz/}G \cdot \text{1G/cm} \cdot 0.235 \times 10^{-3} \text{s}
= 1 \text{cm}^3
\]

\[ k_x(t_1) \]

\[ k_y(t_2) \]

\[ 1 \text{ cm} \]