Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2010
X-Rays Lecture 1

EM spectrum

Tungsten filament heated to about 2200 C leading to thermionic emission of electrons.

Usually tungsten is used for anode
Molybdenum for mammography

Suetens 2002
**X-Ray Production**

Collisional transfers

Radiative transfers

Prince and Links 2005

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**Interaction with Matter**

Typical energy range for diagnostic x-rays is below 200 keV. The two most important types of interaction are photoelectric absorption and Compton scattering.

Photoelectric effect dominates at low x-ray energies and high atomic numbers, \( E_{\text{e}^-} = h\nu - E_B \)

Compton scattering dominates at high x-ray energies and low atomic numbers, not much contrast

http://www.eee.ntu.ac.uk/research/vision/andreas

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**X-Ray Imaging Chain**

low-energy absorbing filter

collimating scatter grid

x-ray source

collimator

detector

Reduces effects of Compton scattering

Suetses 2002
Silver halide crystals absorb optical energy. After development, crystals that have absorbed enough energy are converted to metallic silver and look dark on the screen. Thus, parts in the object that attenuate the x-rays will look brighter.

Intensifying Screen

X-Ray w/ Contrast Agents

Angiogram using an iodine-based contrast agent. K-edge of iodine is 33.2 keV

Barium Sulfate
K-edge of Barium is 37.4 keV

Suetens 2002
Suetens 2002
**Intensity**

\[ I = E \phi \]

- \( I \): Energy
- \( E \): Photon flux rate
- \( \phi \): Number of photons
- \( N \): Unit Area
- \( \Delta t \): Unit Time

**Intensity**

\[ \phi = \int_{E_0}^{\infty} S(E')dE' \]

- \( S(E') \): X-ray spectrum

\[ I = \int_{E_0}^{\infty} S(E')E'dE' \]

**Attenuation**

\[ n = \mu N \Delta x \quad \text{photons per unit length} \]

\[ \mu = \frac{n}{N} \quad \text{fraction of photons lost per unit length} \]

\[ \Delta N = -n \quad \frac{dN}{dx} = -\mu N \quad N(x) = N_0 e^{-\mu x} \]

For single-energy x-rays passing through a homogenous object:

\[ I_{out} = I_{in} \exp(-\mu d) \]

- \( I_{out} \): Linear attenuation coefficient

**Attenuation**

\[ I(\Delta x) = I_0 e^{-\mu \Delta x} \]
Attenuation

Inhomogeneous Slab

\[
\frac{dN}{dx} = -\mu(x)N \\
N(x) = N_0 \exp \left( -\int_0^x \mu(x')dx' \right) \\
I(x) = I_0 \exp \left( -\int_0^x \mu(x')dx' \right)
\]

Attenuation depends on energy, so also need to integrate over energies

\[
I(x) = \int_{0}^{\infty} S_0(\varepsilon')\varepsilon' \exp \left\{ -\int_{0}^{x} \mu(x';\varepsilon')dx' \right\} d\varepsilon'
\]

Half Value Layer

<table>
<thead>
<tr>
<th>X-ray energy (keV)</th>
<th>HVL, muscle (cm)</th>
<th>HVL Bone (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>30</td>
<td>1.8</td>
<td>0.4</td>
</tr>
<tr>
<td>50</td>
<td>3.0</td>
<td>1.2</td>
</tr>
<tr>
<td>100</td>
<td>3.9</td>
<td>2.3</td>
</tr>
<tr>
<td>150</td>
<td>4.5</td>
<td>2.8</td>
</tr>
</tbody>
</table>

In chest radiography, about 90% of x-rays are absorbed by body. Average energy from a tungsten source is 68 keV. However, many lower energy beams are absorbed by tissue, so average energy is higher. This is referred to as beam-hardening, and reduces the contrast.

Attenuation

More Attenuation

\[
\begin{array}{c}
\text{Photoelectric effect domi} \\
\text{nates}
\end{array}
\]

Compton Scattering dominates

Less Attenuation

Contrast

Bushberg et al 2001

Values from Webb 2003

Photons Absorption

Bushberg et al 2001
Contrast

\[ A = N_0 e^{-\mu x} \]
\[ B = N_0 e^{-\mu (x + z)} \]
\[ C_s = A - B \]
\[ A = N_0 e^{-\mu x} \]
\[ B = N_0 e^{-\mu (x + z)} \]
\[ C_s = \frac{A - B}{A} = \frac{N_0 e^{-\mu x} - N_0 e^{-\mu (x + z)}}{N_0 e^{-\mu x}} = 1 - e^{-\mu z} \]

Subject Contrast

X-Ray Imaging Geometry

**Inverse Square Law**

Inverse Square Law

\[ I_0 = \frac{I_x}{4\pi d^2} \]
\[ I_x(x, y) = \frac{I_0}{4\pi r^2} \]
\[ = \frac{I_0 d^2}{r^2} = I_0 \cos^2 \theta \]

Bushberg et al 2001

Prince and Links 2005

Prince and Links 2005
**Obliquity Factor**

\[ I_d(x, y) = I_0 \cos \theta \]

**X-Ray Imaging Geometry**

Beam Divergence and Flat Panel

\[ I_r = I_0 \cos^3 \theta \]

Example: Chest x-ray at 2 yards with 14x17 inch film.

Question: What is the smallest ratio \( \frac{I_r}{I_0} \) across the film?

\[ r_d = \sqrt{7^2 + 8.5^2} = 11 \]

\[ \cos \theta = \frac{d}{\sqrt{r^2 + d^2}} = 0.989 \]

\[ \frac{I_r}{I_0} = \cos^3 \theta = 0.966 \]

**Anode Heel Effect**

**Compensation Filters**

[Link: http://www.animalinsides.com/radphys/chapters/Lec2.pdf]

[Prince and Links 2005]
Path Length

\[ L' = L / \cos \theta \]

\[ I_d(x, y) = I_0 \cos^3 \theta \exp(-\mu L / \cos \theta) \]

Magnification of Object

\[ M(z) = \frac{d}{z} \]

Source to Image Distance (SID)

Source to Object Distance (SOD)

X-Ray Imaging Equation

At \( z = d \) there is no magnification, so

\[ I_d(x, y) = I_0 \cos^3 \theta \cdot \exp \left( - \int_{L_{x,y}} \mu(x) ds / \cos \theta \right) \]

\[ = I_0 \cos^3 \theta \cdot t_z(x, y) \]

where \( t_z(x, y) \) is the transmittivity of the object at distance \( z \)

In general, with magnification

\[ I_d(x, y) = I_0 \cos^3 \theta \cdot t_z(x/M(z), y/M(z)) \]

Magnification of Object

\[ M = 1: \ I(x, y) = t(x, y) \]

\[ M = 2: \ I(x, y) = t(x/2, y/2) \]

In general, \( I(x, y) = t(x/M(z), y/M(z)) \)
Image of a point object

\[ I_d(x, y) = ks(x/m, y/m) \]

\[ \int \int ks(x/m(z), y/m(z))dxdy = \text{constant} \]

\[ \Rightarrow k = \frac{1}{m^2(z)} \]

\[ I_d(x, y) = \lim_{m \to 0} \frac{s(x/m, y/m)}{m^2} = \delta(x, y) \]

Image of arbitrary object

\[ s(x, y) \quad t(x, y) \]

\[ \lim_{m \to 0} I_d(x, y) = t(x, y) \]

\[ s(x, y) \quad t(x, y) \]

\[ m=1 \quad I_d(x, y) = ??? \]

\[ I_d(x, y) = \frac{\cos^3 \theta}{4\pi d^2 m^2} s(x/m, y/m) * * t(x/M, y/M) \]
Convolution

$$I(x, y) = \frac{\cos^3 \theta}{4 \pi d^2 m} s(x/m, y/m) * t(x/M, y/M) * h(x, y)$$

Macovski 1983

Signals and Images

Discrete-time/space signal image: continuous valued function with a discrete time/space index, denoted as $s[n]$ for 1D, $s[m,n]$ for 2D, etc.

Continuous-time/space signal image: continuous valued function with a continuous time/space index, denoted as $s(t)$ or $s(x)$ for 1D, $s(x,y)$ for 2D, etc.
Kronecker Delta Function

\[ \delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \text{otherwise} \end{cases} \]

Discrete Signal Expansion

\[ g[n] = \sum_{k=-\infty}^{\infty} g[k] \delta[n-k] \]

\[ g[m,n] = \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} g[k,l] \delta[m-k,n-l] \]

2D Signal

\[ \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix} \]
Image Decomposition

\[
g[m,n] = a \delta[m,n] + b \delta[m-1,n] + c \delta[m-1,n-1] + d \delta[m-1,n-1]
= \sum_{k=0}^{l} \sum_{l=0}^{l} g[k,l] \delta[m-k,n-l]
\]

Dirac Delta Function

Notation:

\(\delta(x)\) - 1D Dirac Delta Function
\(\delta(x,y)\) or \(\delta(x,y)\) - 2D Dirac Delta Function
\(\delta(x,y,z)\) or \(\delta(x,y,z)\) - 3D Dirac Delta Function
\(\delta(\vec{r})\) - N Dimensional Dirac Delta Function

1D Dirac Delta Function

\(\delta(x) = 0\) when \(x \neq 0\) and \(\int_{\infty}^{-\infty} \delta(x) \, dx = 1\)
Can interpret the integral as a limit of the integral of an ordinary function that is shrinking in width and growing in height, while maintaining a constant area. For example, we can use a shrinking rectangle function such that \(\int_{\infty}^{-\infty} \delta(x) \, dx = \lim_{\tau \to 0} \int_{\tau}^{\infty} \Pi(x/\tau) \, dx\).

2D Dirac Delta Function

\(\delta(x,y) = 0\) when \(x^2 + y^2 \neq 0\) and \(\iint_{\infty}^{-\infty} \delta(x,y) \, dx \, dy = 1\)
where we can consider the limit of the integral of an ordinary 2D function that is shrinking in width but increasing in height while maintaining constant area.
\(\iint_{\infty}^{-\infty} \delta(x,y) \, dx \, dy = \lim_{\tau \to 0} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tau^2 H(x/\tau, y/\tau) \, dx \, dy\).
Useful fact: \(\delta(x,y) = \delta(x) \delta(y)\)
Generalized Functions

Dirac delta functions are not ordinary functions that are defined by their value at each point. Instead, they are generalized functions that are defined by what they do underneath an integral.

The most important property of the Dirac delta is the sifting property

\[ \int_{-\infty}^{\infty} \delta(x - x_0)g(x)dx = g(x_0) \]

where \( g(x) \) is a smooth function. This sifting property can be understood by considering the limiting case

\[ \lim_{\tau \to 0} \int_{-\infty}^{\infty} \tau \Pi(x/\tau)g(x)dx = g(x_0) \]

From the sifting property, we can write a 1D function as

\[ g(x) = \int_{-\infty}^{\infty} g(\xi)\delta(x - \xi)d\xi. \]

To gain intuition, consider the approximation

\[ g(x) = \sum_{n=1}^{\infty} g(n\Delta x)\frac{1}{\Delta x} \int_{-\infty}^{\infty} \Pi(x - n\Delta x)dx. \]

Similarly, we can write a 2D function as

\[ g(x,y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi,\eta)\delta(x - \xi, y - \eta)d\xi d\eta. \]

To gain intuition, consider the approximation

\[ g(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} g(n\Delta x, m\Delta y)\frac{1}{\Delta x} \int_{-\infty}^{\infty} \Pi(x - n\Delta x)dx \frac{1}{\Delta y} \int_{-\infty}^{\infty} \Pi(y - m\Delta y)dy. \]

Intuition: the impulse response is the response of a system to an input of infinitesimal width and unit area.

Since any input can be thought of as the weighted sum of impulses, a linear system is characterized by its impulse response(s).
Impulse Response

The impulse response characterizes the response of a system over all space to a Dirac delta impulse function at a certain location.

\[ h(x_1; \xi) = L[\delta(x_1 - \xi)] \quad \text{1D Impulse Response} \]
\[ h(x_1, y_1; \xi, \eta) = L[\delta(x_1 - \xi, y_1 - \eta)] \quad \text{2D Impulse Response} \]

Impulse at \( \xi, \eta \)

Full Width Half Maximum (FWHM) is a measure of resolution.