Linearity (Addition)

A system $R$ is linear if for two inputs $I_1(x,y)$ and $I_2(x,y)$ with outputs $R(I_1(x,y))=K_1(x,y)$ and $R(I_2(x,y))=K_2(x,y)$

the response to the weighted sum of inputs is the weighted sum of outputs:

$$R(a_1I_1(x,y)+a_2I_2(x,y))=a_1K_1(x,y)+a_2K_2(x,y)$$

Linearity (Scaling)

$$R(a_1I_1(x,y))=a_1K_1(x,y)$$

$$R(a_1I_1(x,y))+R(a_2I_2(x,y))=a_1K_1(x,y)+a_2K_2(x,y)$$
Example

Are these linear systems?

\[
g(x,y) \quad + \quad g(x,y) + 10 \\
\]

\[
g(x,y) \quad \times \quad 10g(x,y) \\
\]

Move up
By 1

Move right
By 1

g(x-1,y-1)

Superposition Integral

What is the response to an arbitrary function \( g(x_1,y_1) \)?

Write \( g(x_1,y_1) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi,\eta) \delta(x_1 - \xi, y_1 - \eta) d\xi d\eta \).

The response is given by

\[
l(x_2,y_2) = L[g(x_1,y_1)] \]

\[
= L\left[ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi,\eta) \delta(x_1 - \xi, y_1 - \eta) d\xi d\eta \right] \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi,\eta) L[\delta(x_1 - \xi, y_1 - \eta)] d\xi d\eta \\
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi,\eta) h(x_2,y_2;\xi,\eta) d\xi d\eta \\
\]

Space Invariance

If a system is space invariant, the impulse response depends only on the difference between the output coordinates and the position of the impulse and is given by

\[
h(x_2,y_2;\xi,\eta) = h(x_2 - \xi, y_2 - \eta) \\
\]

Superposition

\[
g(m) = g[0]\delta[m] + g[1]\delta[m - 1] + g[2]\delta[m - 2] \]

\[
h[m',k] = L[\delta[m - k]] \\
\]

\[
y[m'] = L[g[m]] \]

\[
= L[g[0]\delta[m] + g[1]\delta[m - 1] + g[2]\delta[m - 2]] \\
= L[g[0]L[\delta[m]] + L[g[1]\delta[m - 1]] + L[g[2]\delta[m - 2]]] \\
= g[0]L[\delta[m]] + g[1]L[\delta[m - 1]] + g[2]L[\delta[m - 2]] \\
= g[0]h[m',0] + g[1]h[m',1] + g[2]h[m',2] \\
= \sum_{k=0}^{2} g[k] h[m',k] \\
\]
Pinhole Magnification Example

___, the pinhole system ___ space invariant.

1D Convolution

\[ I(x) = \int_{-\infty}^{\infty} g(\xi) h(x; \xi) d\xi \]
\[ = \int_{-\infty}^{\infty} g(\xi) h(x - \xi) d\xi \]
\[ = g(x) * h(x) \]

Useful fact:
\[ g(x) * \delta(x - \Delta) = \int_{-\infty}^{\infty} g(\xi) \delta(x - \xi - \Delta) d\xi \]
\[ = g(x - \Delta) \]

Convolution

\[ g[m] = g[0]\delta(m) + g[1]\delta(m-1) + g[2]\delta(m-2) \]
\[ h[m', k] = L[\delta(m - k)] = h[m' - k] \]
\[ y[m'] = L[g[m]] \]
\[ = L[g[0]\delta(m)] + L[g[1]\delta(m-1)] + L[g[2]\delta(m-2)] \]
\[ = g[0]L[\delta(m)] + g[1]L[\delta(m-1)] + g[2]L[\delta(m-2)] \]
\[ = g[0]h[m' - 0] + g[1]h[m' - 1] + g[2]h[m' - 2] \]
\[ = \sum_{k=0}^{2} g[k]h[m' - k] \]

2D Convolution

For a space invariant linear system, the superposition integral becomes a convolution integral.

\[ I(x_2, y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi, \eta) h(x_2 - \xi, y_2 - \eta) d\xi d\eta \]
\[ = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi, \eta) h(x_2 - \xi, y_2 - \eta) d\xi d\eta \]
\[ = g(x_2, y_2) ** h(x_2, y_2) \]

where ** denotes 2D convolution. This will sometimes be abbreviated as *, e.g. \( I(x_2, y_2) = g(x_2, y_2) * h(x_2, y_2) \).
**Rectangle Function**

\[ \Pi(x) = \begin{cases} 0 & |x| > \frac{1}{2} \\ 1 & |x| \leq \frac{1}{2} \end{cases} \]

Also called \( \text{rect}(x) \)

\[ \Pi(x,y) = \Pi(x)\Pi(y) \]

**1D Convolution Examples**

\[ g(x) = \delta(x+\frac{1}{2},y) + \delta(x,y) \]

\[ h(x) = \text{rect}(x,y) \]

\[ I(x,y) = g(x) * h(x,y) \]

**2D Convolution Example**
For off-center pinhole object, the shifted source image can be written as
\[ s \left( \frac{x - Mx_0}{m} \right) = s \left( \frac{x}{m} \right) \frac{1}{M} \delta \left( \frac{x - Mx_0}{M} \right) = s(x/m) \ast \left( \frac{x}{M} \right) \]

For the general 2D case, we convolve the magnified object with the impulse response
\[ I(x, y) = t \left( \frac{x}{M}, \frac{y}{M} \right) \ast \ast \frac{1}{m^2} \delta \left( \frac{x}{m}, \frac{y}{m} \right) \]

Note: we have ignored obliquity factors etc.
X-Ray Imaging

\[ m = 1, M = 2 \]

\[ \frac{1}{m} \left( \frac{x}{m} \right) \ast \left( \frac{x}{M} \right) = \text{rect}(x/10) \ast \text{rect}(x/20) \]

= ???

Summary

1. The response to a linear system can be characterized by a spatially varying impulse response and the application of the superposition integral.

2. A shift invariant linear system can be characterized by its impulse response and the application of a convolution integral.