Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2013
MRI Lecture 2

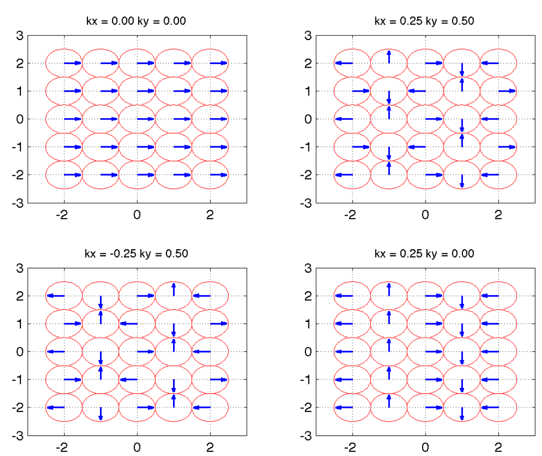
2D Fourier Transform

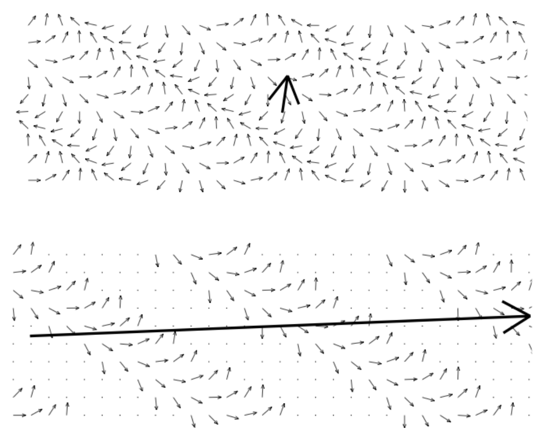
Fourier Transform

$$G(k_x,k_y) = F\left[g(x,y)\right] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)e^{-j2\pi(k_x x + k_y y)}dxdy$$

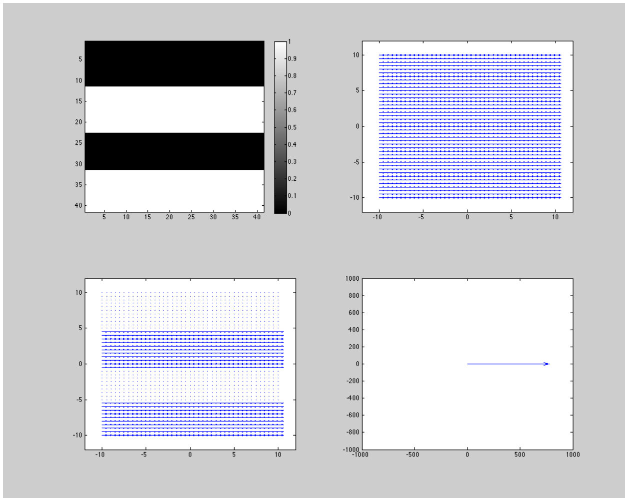
Inverse Fourier Transform

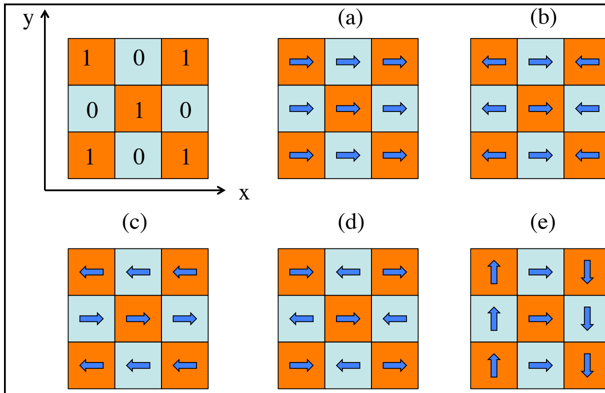
$$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G(k_x,k_y)e^{j2\pi(k_x x + k_y y)}dk_x dk_y$$





Hanson 2009





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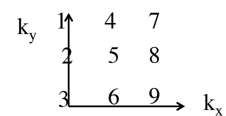


Gradient Fields

Define

$$\vec{G} \equiv G_x\hat{i} + G_y\hat{j} + G_z\hat{k} \qquad \vec{r} \equiv x\hat{i} + y\hat{j} + z\hat{k}$$

So that

$$G_x x + G_y y + G_z z = \vec{G}\cdot\vec{r}$$

Also, let the gradient fields be a function of time. Then the z-directed magnetic field at each point in the volume is given by :

$$B_z(\vec{r},t) = B_0 + \vec{G}(t)\cdot\vec{r}$$

Static Gradient Fields

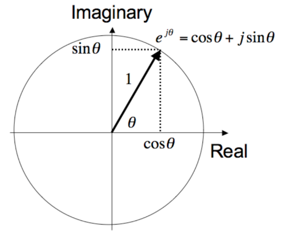
In a uniform magnetic field, the transverse magnetization is given by:

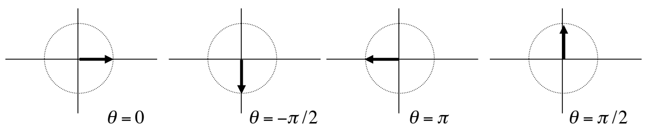
$$M(t) = M(0)e^{-j\omega_0 t}e^{-t/T_2}$$

In the presence of non time-varying gradients we have

$$\begin{aligned} M(\vec{r}) &= M(\vec{r},0)e^{-j\gamma B_z(\vec{r})t}e^{-t/T_2(\vec{r})} \\ &= M(\vec{r},0)e^{-j\gamma(B_0+\vec{G}\cdot\vec{r})t}e^{-t/T_2(\vec{r})} \\ &= M(\vec{r},0)e^{-j\omega_0 t}e^{-j\gamma\vec{G}\cdot\vec{r}t}e^{-t/T_2(\vec{r})} \end{aligned}$$

Phasors





Phase

Phase = angle of the magnetization phasor
Frequency = rate of change of angle (e.g. radians/sec)
Phase = time integral of frequency

$$\varphi(\vec{r},t) = -\int_0^t \omega(\vec{r},\tau)d\tau$$
$$= -\omega_0 t + \Delta\varphi(\vec{r},t)$$

Where the incremental phase due to the gradients is

$$\Delta\varphi(\vec{r},t) = -\int_0^t \Delta\omega(\vec{r},\tau)d\tau$$
$$= -\int_0^t \gamma\vec{G}(\vec{r},\tau)\cdot\vec{r}\,d\tau$$

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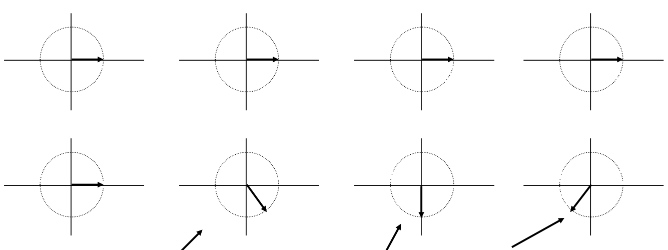
Phase with constant gradient



$$\Delta\varphi(\vec{r},t_1) = -\int_0^{t_1} \Delta\omega(\vec{r},\tau)d\tau$$

$$\Delta\varphi(\vec{r},t_2) = -\int_0^{t_2} \Delta\omega(\vec{r},\tau)d\tau$$
$$= -\Delta\omega(\vec{r})t_2$$
if $\Delta\omega$ is non - time varying.

$$\Delta\varphi(\vec{r},t_3) = -\int_0^{t_3} \Delta\omega(\vec{r},\tau)d\tau$$

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Time-Varying Gradient Fields

In the presence of time-varying gradients the frequency as a function of space and time is:

$$\omega(\vec{r},t) = \gamma B_z(\vec{r},t)$$
$$= \gamma B_0 + \gamma\vec{G}(t)\cdot\vec{r}$$
$$= \omega_0 + \Delta\omega(\vec{r},t)$$

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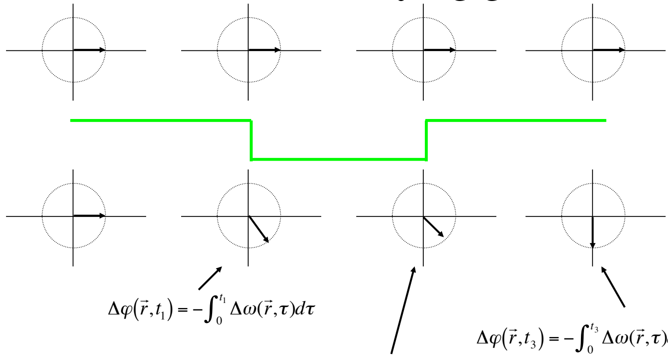
Phase with time-varying gradient



$$\Delta\varphi(\vec{r},t_1) = -\int_0^{t_1}\Delta\omega(\vec{r},\tau)d\tau$$

$$\Delta\varphi(\vec{r},t_3) = -\int_0^{t_3}\Delta\omega(\vec{r},\tau)d\tau$$

$$\Delta\varphi(\vec{r},t_2) = -\int_0^{t_2}\Delta\omega(\vec{r},\tau)d\tau = -\int_0^{t_1}\Delta\omega(\vec{r},\tau)d\tau + 0$$

Time-Varying Gradient Fields

The transverse magnetization is then given by

$$\begin{aligned} M(\vec{r},t) &= M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{\varphi(\vec{r},t)} \\ &= M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\int_o^t \Delta\omega(\vec{r},t)d\tau\right) \\ &= M(\vec{r},0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right) \end{aligned}$$

Signal Equation

Signal from a volume

$$\begin{aligned} s_r(t) &= \int_V M(\vec{r},t)dV \\ &= \int_x\int_y\int_z M(x,y,z,0)e^{-t/T_2(\vec{r})}e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)dxdydz \end{aligned}$$

For now, consider signal from a slice along z and drop the T_2 term. Define $m(x,y) \equiv \int_{z_0-\Delta z/2}^{z_0+\Delta z/2} M(\vec{r},t)dz$

To obtain

$$s_r(t) = \int_x\int_y m(x,y)e^{-j\omega_0 t}\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)dxdy$$

Signal Equation

Demodulate the signal to obtain

$$\begin{aligned} s(t) &= e^{j\omega_0 t}s_r(t) \\ &= \int_x\int_y m(x,y)\exp\left(-j\gamma\int_o^t \vec{G}(\tau)\cdot\vec{r}\,d\tau\right)dxdy \\ &= \int_x\int_y m(x,y)\exp\left(-j\gamma\int_o^t \left[G_x(\tau)x + G_y(\tau)y\right]d\tau\right)dxdy \\ &= \int_x\int_y m(x,y)\exp\left(-j2\pi\left(k_x(t)x + k_y(t)y\right)\right)dxdy \end{aligned}$$

Where

$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

$$k_y(t) = \frac{\gamma}{2\pi}\int_0^t G_y(\tau)d\tau$$

MR signal is Fourier Transform

$$s(t) = \int_x \int_y m(x,y)\exp\left(-j2\pi\left(k_x(t)x + k_y(t)y\right)\right)dxdy$$
$$= M\left(k_x(t), k_y(t)\right)$$
$$= F\left[m(x,y)\right]\Big|_{k_x(t),k_y(t)}$$

Recap

- Frequency = rate of change of phase.
- Higher magnetic field -> higher Larmor frequency -> phase changes more rapidly with time.
- With a constant gradient G_x, spins at different x locations precess at different frequencies -> spins at greater x-values change phase more rapidly.
- With a constant gradient, distribution of phases across x locations changes with time. (phase modulation)
- More rapid change of phase with x -> higher spatial frequency k_x

K-space

At each point in time, the received signal is the Fourier transform of the object

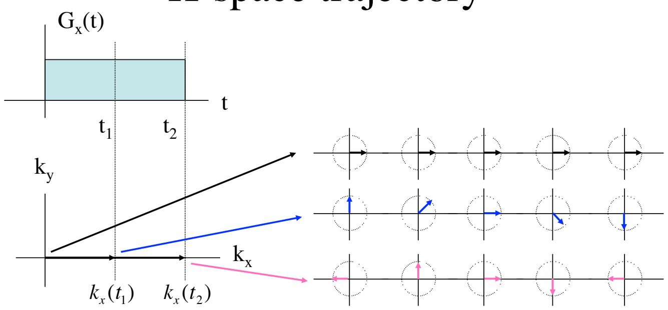
$$s(t) = M\left(k_x(t), k_y(t)\right) = F\left[m(x,y)\right]\Big|_{k_x(t),k_y(t)}$$

evaluated at the spatial frequencies:

$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$
$$k_y(t) = \frac{\gamma}{2\pi}\int_0^t G_y(\tau)d\tau$$

Thus, the gradients control our position in k-space. The design of an MRI pulse sequence requires us to efficiently cover enough of k-space to form our image.

K-space trajectory



$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

Units

Spatial frequencies (k_x, k_y) have units of 1/distance. Most commonly, 1/cm

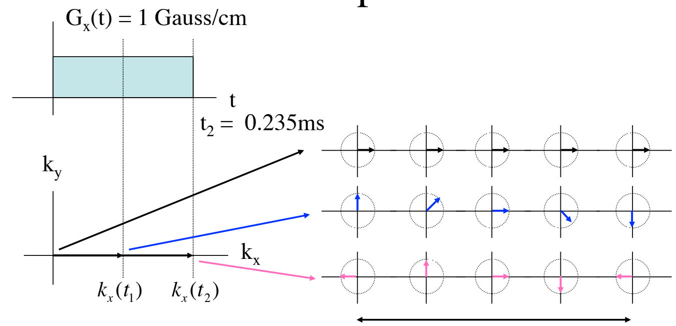
Gradient strengths have units of (magnetic field)/ distance. Most commonly G/cm or mT/m

$\gamma/(2\pi)$ has units of Hz/G or Hz/Tesla.

$$k_x(t) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$

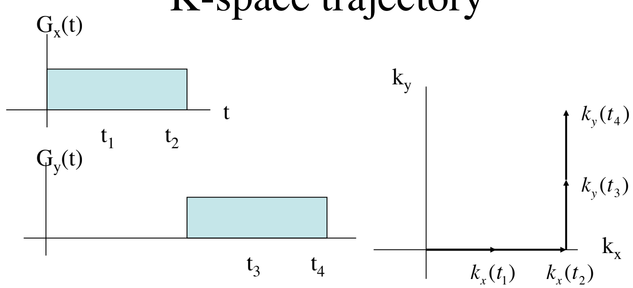
$$= [Hz/Gauss][Gauss/cm][\sec]$$
$$= [1/cm]$$

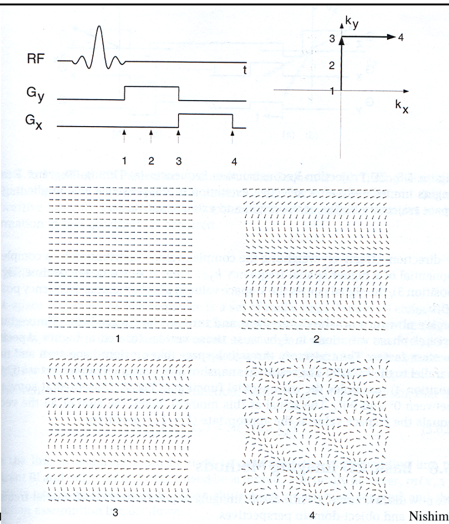
Example



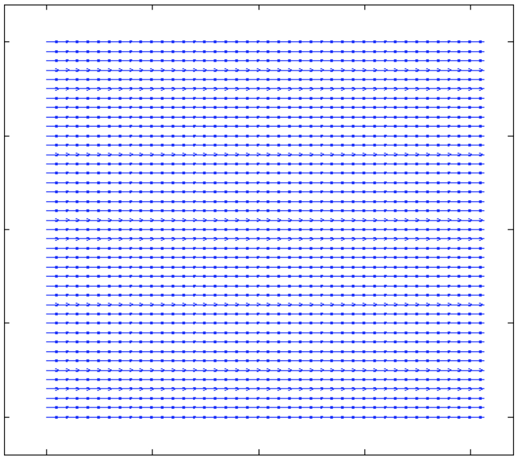
$$k_x(t_2) = \frac{\gamma}{2\pi}\int_0^t G_x(\tau)d\tau$$
$$= 4257 Hz/G \cdot 1G/cm \cdot 0.235\times 10^{-3} s$$
$$= 1\ cm^{-1}$$

K-space trajectory

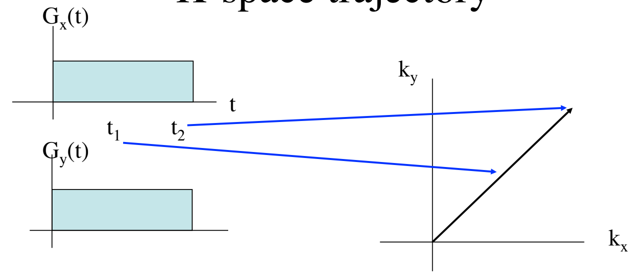




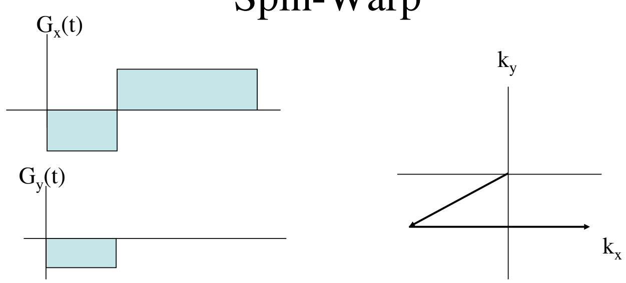
Nishimura 1996



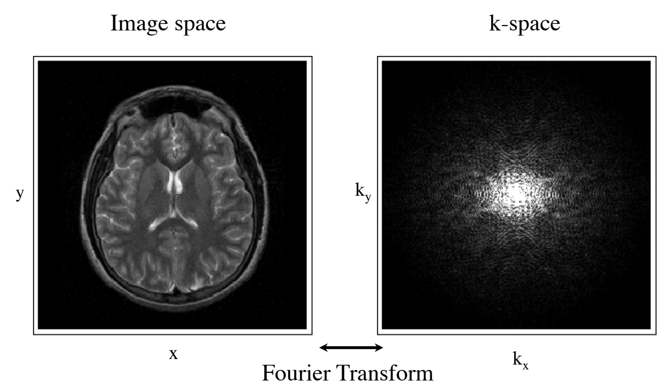
K-space trajectory

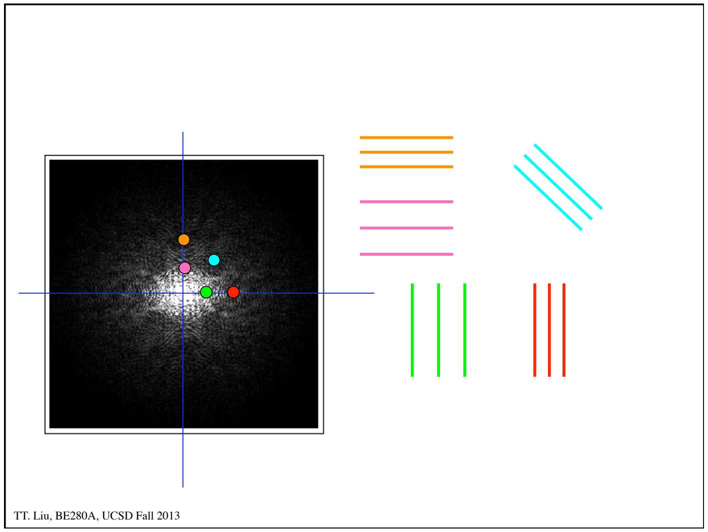


Spin-Warp



k-space

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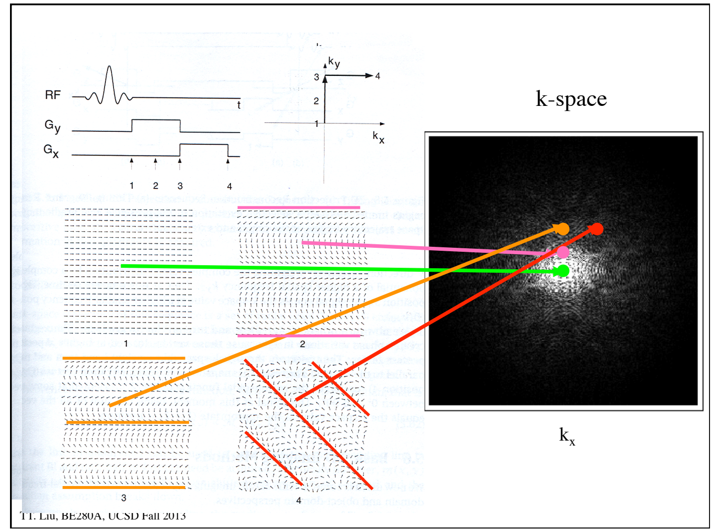
RF
G_y
G_x
t
1 2 3 4
k_y
k_x
k-space
k_x
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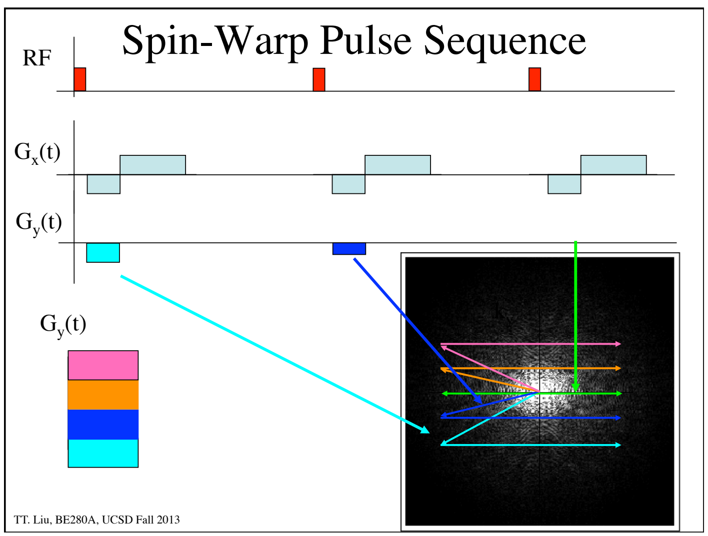
Spin-Warp Pulse Sequence
RF
$G_x(t)$
$G_y(t)$
$G_y(t)$
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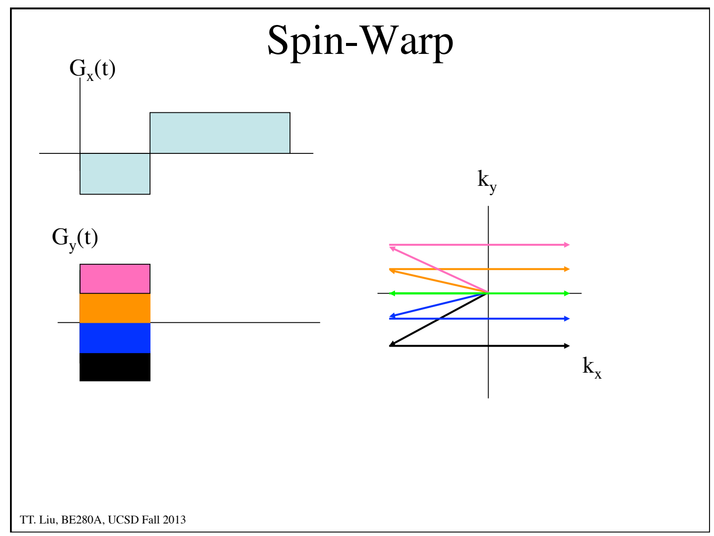
Spin-Warp
$G_x(t)$
$G_y(t)$
k_y
k_x
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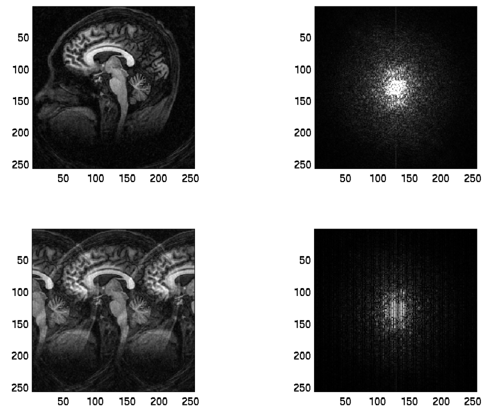
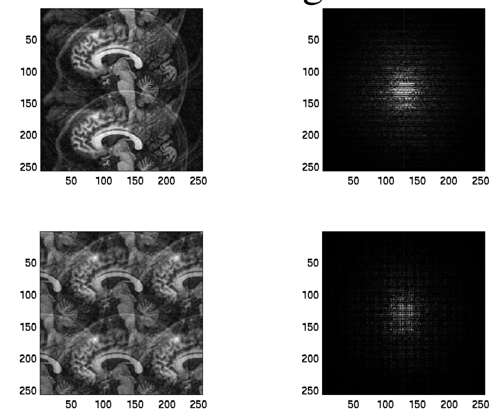
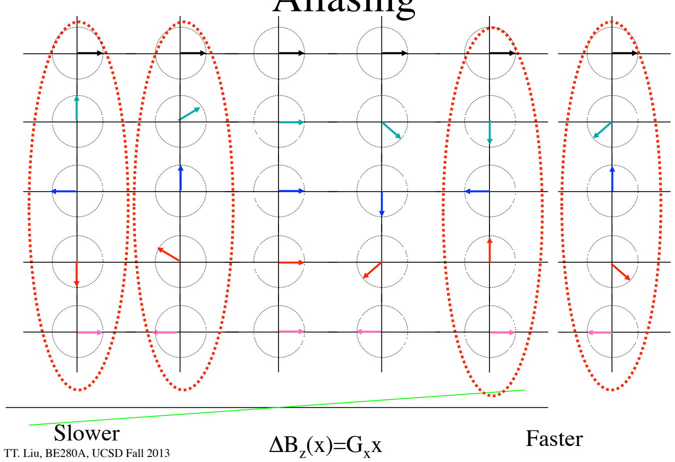
Sampling in k-space



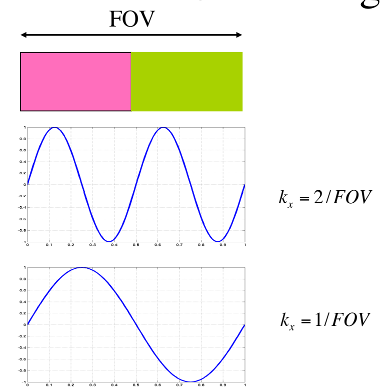
Aliasing



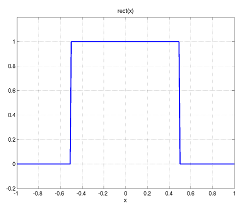
Aliasing



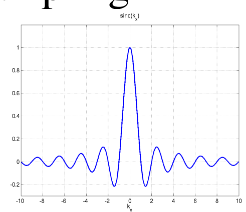
Intuitive view of Aliasing



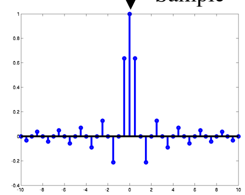
Fourier Sampling



$\rightarrow$ F



$\downarrow$ Sample

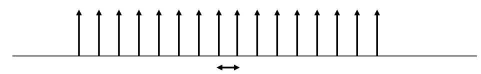


Instead of sampling the signal, we sample its Fourier Transform

??? $\leftarrow$ F^{-1}

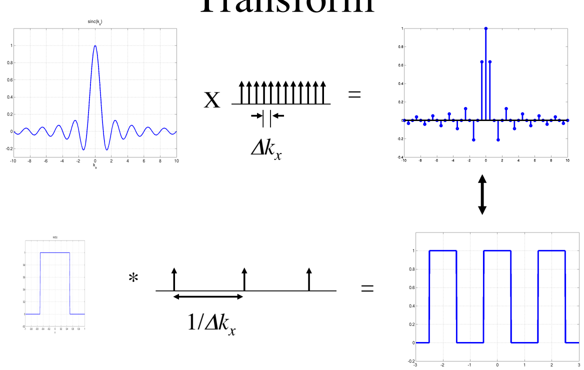
Fourier Sampling

$(1/\Delta k_x)\ comb(k_x/\Delta k_x)$

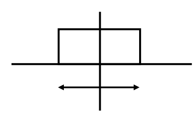


$$
\begin{aligned}
G_S(k_x) &= G(k_x)\frac{1}{\Delta k_x} comb\left(\frac{k_x}{\Delta k_x}\right) \\
&= G(k_x)\sum_{n=-\infty}^{\infty}\delta(k_x - n\Delta k_x) \\
&= \sum_{n=-\infty}^{\infty} G(n\Delta k_x)\delta(k_x - n\Delta k_x)
\end{aligned}
$$

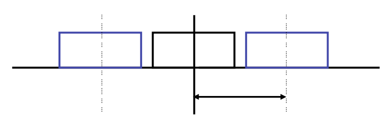
Fourier Sampling -- Inverse Transform



Nyquist Condition

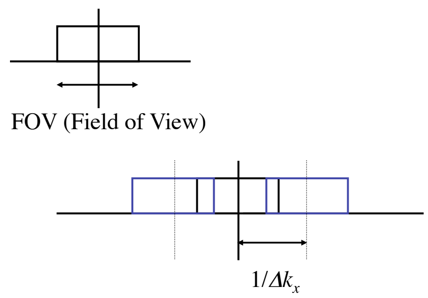


FOV (Field of View)



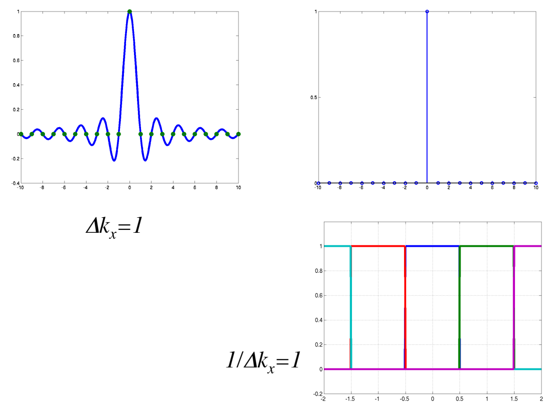
To avoid overlap, $1/\Delta k_x >$ FOV, or equivalently, $\Delta k_x < 1/$FOV

Aliasing



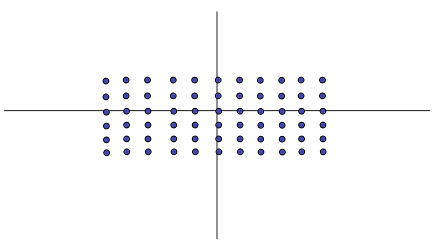
Aliasing occurs when $1/\Delta k_x <$ FOV

Aliasing Example



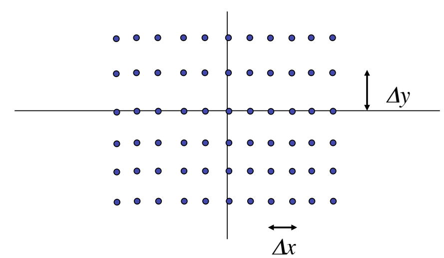
2D Comb Function

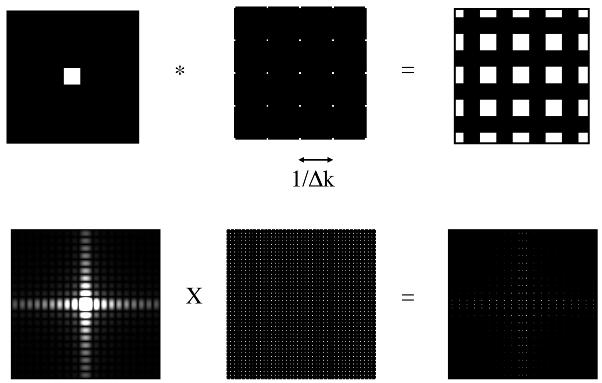
$$\begin{aligned} comb(x,y) &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m, y-n) \\ &= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m)\delta(y-n) \\ &= comb(x)comb(y) \end{aligned}$$



Scaled 2D Comb Function

$$\begin{aligned} comb(x/\Delta x, y/\Delta y) &= comb(x/\Delta x)comb(y/\Delta y) \\ &= \Delta x \Delta y \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(x-m\Delta x)\delta(y-n\Delta y) \end{aligned}$$





2D k-space sampling

$$G_S(k_x,k_y) = G(k_x,k_y)\frac{1}{\Delta k_x \Delta k_y} comb\left(\frac{k_x}{\Delta k_x},\frac{k_y}{\Delta k_y}\right)$$

$$= G(k_x,k_y)\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}\delta(k_x - m\Delta k_x, k_y - n\Delta k_y)$$

$$= \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} G(m\Delta k_x, n\Delta k_y)\delta(k_x - m\Delta k_x, k_y - n\Delta k_y)$$

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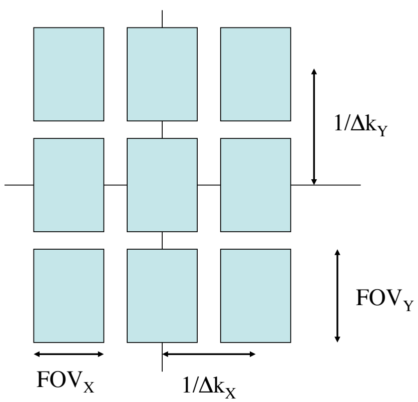
Nyquist Conditions



$1/\Delta k_Y > FOV_Y$

$1/\Delta k_X > FOV_X$

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