Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2014
MRI Lecture 4

K-space

At each point in time, the received signal is the Fourier transform of the object

\[ s(t) = M(k_x(t), k_y(t)) = F[m(x, y)]_{k_x(t), k_y(t)} \]

evaluated at the spatial frequencies:

\[ k_x(t) = \frac{\gamma}{2\pi} \int G_x(r)dr \]
\[ k_y(t) = \frac{\gamma}{2\pi} \int G_y(r)dr \]

Thus, the gradients control our position in k-space. The design of an MRI pulse sequence requires us to efficiently cover enough of k-space to form our image.
Nyquist Conditions

\[ \frac{1}{\Delta k_x} > \text{FOV}_x \]
\[ \frac{1}{\Delta k_y} > \text{FOV}_y \]

1/\\Delta k_x > \text{FOV}_X \]
1/\\Delta k_y > \text{FOV}_Y \]

Resolution and spatial frequency

With a window of width \( W_x \), the highest spatial frequency is \( W_x / 2 \).
This corresponds to a spatial period of \( 2/W_x \).

\[ \frac{1}{W_x} = \text{Effective Width} \]

[Diagram of resolution and spatial frequency]

Sampling and Windowing

\[ \Delta k_x < 1/(\text{FOV}_x) \]
\[ \Delta k_y < 1/(\text{FOV}_y) \]

\[ W_{k_x} > 1/\delta_x \]
\[ W_{k_y} > 1/\delta_y \]

[Diagram of sampling and windowing]
**Example**

*Goal:*

\[ \text{FOV}_x = \text{FOV}_y = 25.6 \text{ cm} \]

\[ \delta_x = \delta_y = 0.1 \text{ cm} \]

**Readout Gradient:**

\[ \text{FOV}_x = \frac{1}{2\pi} \frac{\Delta \text{ADC}}{\Delta t} \]

Assume \( \Delta t = 32 \text{ usec} \)

\[ G_x = \frac{1}{\text{FOV}_x} \frac{\Delta \text{ADC}}{\Delta t} = \frac{1}{25.6 \text{ cm}} \frac{425.7 \times 10^3 \text{ T} \cdot \text{s}}{32 \times 10^{-6} \text{s}} \]

\[ = 2.8675 \times 10^8 \text{T/cm} \]

\[ = 0.28675 \text{ G/cm} \]

\[ N_{\text{read}} = \frac{\text{FOV}_x \delta_x}{\Delta t} = 256 \]

where \( G_x = 1 \times 10^{-5} \text{ Tesla} \)

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**Phase-Encode Gradient:**

\[ \text{FOV}_y = \frac{1}{2\pi} \frac{\Delta \tau_y}{\Delta t} \]

Assume \( \Delta t = 4.096 \text{ usec} \)

\[ G_y = \frac{1}{\text{FOV}_y} \frac{\Delta \tau_y}{\Delta t} = \frac{1}{25.6 \text{ cm}} \frac{425.7 \times 10^3 \text{ T} \cdot \text{s}}{4.096 \times 10^{-6} \text{s}} \]

\[ = 2.2402 \times 10^8 \text{T/cm} \]

\[ = 0.00224 \text{ G/cm} \]

\[ N_y = \frac{\text{FOV}_y \delta_y}{\Delta t} = 256 \]
Example
Consider the k-space trajectory shown below. ADC samples are acquired at the points shown with Δ = 10 µsec. The desired FOV (both x and y) is 10 cm and the desired resolution (both x and y) is 2.5 cm. Draw the gradient waveforms required to achieve the k-space trajectory. Label the waveform with the gradient amplitudes required to achieve the desired FOV and resolution. Also, make sure to label the time axis correctly. Assume γ/(2π) = 4000 Hz / G

Sampling
In practice, an even number (typically power of 2) sample is usually taken in each direction to take advantage of the Fast Fourier Transform (FFT) for reconstruction.

Gibbs Artifact
256x256 image
256x128 image

Apodization
Hanning Window
rect(k_x)

0.5sinc(x)+0.25sinc(x-1)+0.25sinc(x+1)
Temporal filtering in the readout direction limits the readout FOV. So there should never be aliasing in the readout direction.

Lowpass filter in the readout direction to prevent aliasing.

From Buxton 2002
Free Induction Decay (FID)

An excitation pulse rotates the magnetization vector away from its equilibrium state (purely longitudinal). The resulting vector has both longitudinal $M_z$ and transverse $M_{xy}$ components.

Due to thermal interactions, the magnetization will return to its equilibrium state with characteristic time constants.

$T_1$ spin-lattice time constant, return to equilibrium of $M_z$

$T_2$ spin-spin time constant, return to equilibrium of $M_{xy}$

Relaxation

Longitudinal Relaxation

$$\frac{dM_z}{dt} = -\frac{M_z - M_0}{T_1}$$

After a 90 degree pulse $M_z(t) = M_0(1 - e^{-t/T_1})$

Due to exchange of energy between nuclei and the lattice (thermal vibrations). Process continues until thermal equilibrium as determined by Boltzmann statistics is obtained.

The energy $\Delta E$ required for transitions between down to up spins, increases with field strength, so that $T_1$ increases with $B$.

T1 Values

Image, caption: Nishimura, Fig. 4.2
Transverse Relaxation

\[ \frac{dM_{xy}}{dt} = -\frac{M_{xy}}{T_2} \]

Each spin’s local field is affected by the z-component of the field due to other spins. Thus, the Larmor frequency of each spin will be slightly different. This leads to a dephasing of the transverse magnetization, which is characterized by an exponential decay.

\[ T_2 \] is largely independent of field. \( T_2 \) is short for low frequency fluctuations, such as those associated with slowly tumbling macromolecules.

T2 Relaxation

\[ M_{xy}(t) = M_0 e^{-t/T_2} \]

After a 90 degree excitation

T2 Values

<table>
<thead>
<tr>
<th>Tissue</th>
<th>( T_2 ) (ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>gray matter</td>
<td>100</td>
</tr>
<tr>
<td>white matter</td>
<td>92</td>
</tr>
<tr>
<td>muscle</td>
<td>47</td>
</tr>
<tr>
<td>fat</td>
<td>85</td>
</tr>
<tr>
<td>kidney</td>
<td>58</td>
</tr>
<tr>
<td>liver</td>
<td>43</td>
</tr>
<tr>
<td>CSF</td>
<td>4000</td>
</tr>
</tbody>
</table>

Solids exhibit very short \( T_2 \) relaxation times because there are many low frequency interactions between the immobile spins. On the other hand, liquids show relatively long \( T_2 \) values, because the spins are highly mobile and net fields average out.

Table: adapted from Nishimura, Table 4.2
Example

Questions: How can one achieve T2 weighting? What are the relative T2's of the various tissues?

Precession

\[
\begin{bmatrix}
\frac{dM_x}{dt} \\
\frac{dM_y}{dt} \\
\frac{dM_z}{dt}
\end{bmatrix} = \gamma 
\begin{bmatrix}
0 & B_0 & 0 \\
-\gamma B_0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
\]

Useful to define \( M = M_x + jM_y \)

\[
\frac{dM}{dt} = \frac{d}{dt}(M_x + iM_y)
\]

\[
= -j\gamma B_0 M
\]

Solution is a time-varying phasor

\[
M(t) = M(0)e^{-j\gamma B_0 t} = M(0)e^{-\gamma B_0 t}
\]

Question: which way does this rotate with time?

Bloch Equation

\[
\frac{dM}{dt} = M \times \gamma B - \frac{M_x i + M_y j}{T_2} - \frac{(M_z - M_0)k}{T_1}
\]

Precession

Transverse Relaxation

Longitudinal Relaxation

\( i, j, k \) are unit vectors in the x,y,z directions.

Matrix Form with B=B_0

\[
\begin{bmatrix}
\frac{dM_x}{dt} \\
\frac{dM_y}{dt} \\
\frac{dM_z}{dt}
\end{bmatrix} =
\begin{bmatrix}
-1/T_2 & \gamma B_0 & 0 \\
-\gamma B_0 & 1/T_2 & 0 \\
0 & 0 & -1/T_1
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
M_0/T_1
\end{bmatrix}
\]
Z-component solution

\[ M_z(t) = M_0 + (M_z(0) - M_0)e^{-t/T_1} \]

Saturation Recovery

If \( M_z(0) = 0 \) then \( M_z(t) = M_0(1 - e^{-t/T_1}) \)

Inversion Recovery

If \( M_z(0) = -M_0 \) then \( M_z(t) = M_0(1 - 2e^{-t/T_1}) \)

Transverse Component

\[ M = M_x + jM_y \]

\[ \frac{dM}{dt} = \frac{d}{dt}(M_x + iM_y) = -j(\omega_0 + 1/T_1)M \]

\[ M(t) = M(0)e^{-j\omega_0 t/j^1T_2} \]

Summary

1) Longitudinal component recovers exponentially.
2) Transverse component precesses and decays exponentially.

Fact: Can show that \( T_2 < T_1 \) in order for \( \|M(t)\| \leq M_0 \)

Physically, the mechanisms that give rise to \( T_1 \) relaxation also contribute to transverse \( T_2 \) relaxation.
Static Inhomogeneities

In the ideal situation, the static magnetic field is totally uniform and the reconstructed object is determined solely by the applied gradient fields. In reality, the magnet is not perfect and will not be totally uniform. Part of this can be addressed by additional coils called "shim" coils, and the process of making the field more uniform is called "shimming". In the old days this was done manually, but modern magnets can do this automatically.

In addition to magnet imperfections, most biological samples are inhomogeneous and this will lead to inhomogeneity in the field. This is because, each tissue has different magnetic properties and will distort the field.

Field Inhomogeneities

Signal Decay

The spatial nonuniformity in the field can be modeled by adding an additional term to our signal equation.

$$s(t) = \int V M(\tau, t) dV$$

$$= \int \int M(x, y, z, 0) e^{-i\omega_0 t} e^{-\gamma B_0 \tau} \exp \left( -j \gamma \int \int G(\tau) \cdot \mathbf{r} d\tau \right) dxdydz$$

The effect of this nonuniformity is to cause the spins to dephase with time and thus for the signal to decrease more rapidly. To first order this can be modeled as an additional decay term of the form

$$s(t) = \int \int M(x, y, z, 0) e^{-i\omega_0 t} e^{-\gamma B_0 \tau} \exp \left( -j \gamma \int \int G(\tau) \cdot \mathbf{r} d\tau \right) dxdydz$$
The overall decay has the form:

$$\exp\left(-t/T_2^*\right)$$

where

$$\frac{1}{T_2^*} = \frac{1}{T_2} + \frac{1}{T_2'}$$

Due to random motions of spins. Not reversible.

Due to static inhomogeneities. Reversible with a spin-echo sequence.

Spin Echo

Discovered by Erwin Hahn in 1950.

The spin-echo can refocus the dephasing of spins due to static inhomogeneities. However, there will still be $T_2$ dephasing due to random motion of spins.

"There is nothing that nuclear spins will not do for you, as long as you treat them as human beings." Erwin Hahn

Gradient echo sequences exhibit $T_2^*$ decay.

Gradient echo has $\exp(-TE/T_2^*)$ weighting.

Source: http://mrsrl.stanford.edu/~brian/mri-movies/
Spin Echo

Phase at time $\tau$
\[ \varphi(\tau) = \int_0^\tau -\omega_E(\vec{r}) \, dt = -\omega_E(\vec{r}) \tau \]

Phase after 180 pulse
\[ \varphi(\tau') = \omega_E(\vec{r}) \tau \]

Phase at time $2\tau$
\[ \varphi(2\tau) = -\omega_E(\vec{r}) \tau + \omega_E(\vec{r}) \tau = 0 \]

Spin Echo Pulse Sequence

Image Contrast

Different tissues exhibit different relaxation rates, $T_1$, $T_2$, and $T_2^*$. In addition different tissues can have different densities of protons. By adjusting the pulse sequence, we can create contrast between the tissues. The most basic way of creating contrast is adjusting the two sequence parameters: TE (echo time) and TR (repetition time).
**Saturation Recovery Sequence**

<table>
<thead>
<tr>
<th></th>
<th>90°</th>
<th>TE</th>
<th>90°</th>
<th>TE</th>
<th>90°</th>
</tr>
</thead>
<tbody>
<tr>
<td>TR</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Gradient Echo</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ I(x, y) = \rho(x, y) \left[ 1 - e^{-TR/T_1(x, y)} \right] e^{-TE/T_2^*(x, y)} \]

<table>
<thead>
<tr>
<th></th>
<th>90°</th>
<th>180°</th>
<th>90°</th>
<th>90°</th>
</tr>
</thead>
<tbody>
<tr>
<td>TE</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Spin Echo</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\[ I(x, y) = \rho(x, y) \left[ 1 - e^{-TR/T_1(x, y)} \right] e^{-TE/T_2(x, y)} \]

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**T1-Weighted Scans**

Make TE very short compared to either \(T_2\) or \(T_2^*\). The resultant image has both proton and \(T_1\) weighting.

\[ I(x, y) \approx \rho(x, y) \left[ 1 - e^{-TR/T_1(x, y)} \right] \]

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**T2-Weighted Scans**

Make TR very long compared to \(T_1\) and use a spin-echo pulse sequence. The resultant image has both proton and \(T_2\) weighting.

\[ I(x, y) \approx \rho(x, y) e^{-TE/T_2} \]
T2-Weighted Scans

Proton Density Weighted Scans

Make TR very long compared to $T_1$ and use a very short TE. The resultant image is proton density weighted.

\[ I(x, y) = \rho(x, y) \]

Example

<table>
<thead>
<tr>
<th>T1-weighted</th>
<th>Density-weighted</th>
<th>T2-weighted</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tissue</td>
<td>Proton Density</td>
<td>T1 (ms)</td>
</tr>
<tr>
<td>Csf</td>
<td>1.0</td>
<td>4000</td>
</tr>
<tr>
<td>Gray</td>
<td>0.85</td>
<td>1350</td>
</tr>
<tr>
<td>White</td>
<td>0.7</td>
<td>850</td>
</tr>
</tbody>
</table>

PollEv.com/be280a
a) Which is the most T1 weighted?

b) Which is the most T2 weighted?

c) Which is the most PD weighted?
Inversion Recovery