Overview

Backprojection blurs out the image – need to do a filtered backprojection

Computing Transforms

\[ F(\delta(x)) = \int_{-\infty}^{\infty} \delta(x) e^{-j2\pi k x} dx = 1 \]

\[ F(\delta(x-x_0)) = \int_{-\infty}^{\infty} \delta(x-x_0) e^{-j2\pi k x} dx = e^{-j2\pi k x_0} \]

\[ F(\Pi(x)) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-j2\pi k x} dx = \frac{\sin(\pi k)}{\pi k} = \text{sinc}(k) \]

\[ F(1) = \int_{-\infty}^{\infty} e^{-j2\pi k x} dx = \delta(k) \]

Define \( h(k) = \int_{-\infty}^{\infty} e^{-j2\pi k x} dx \) and see what it does under an integral.

\[
\int_{-\infty}^{\infty} G(k) h(k) dk = \int_{-\infty}^{\infty} G(k) \int_{-\infty}^{\infty} e^{-j2\pi k x} dx dk = \int_{-\infty}^{\infty} g(-x) dx = G(0)
\]

Therefore, \( F(1) = \int_{-\infty}^{\infty} e^{-j2\pi k x} dx = \delta(k) \)
Linearity

The Fourier Transform is linear.

\[ F\{ag(x) + bh(x)\} = aG(k_x) + bH(k_x) \]

\[ F\{ag(x,y) + bh(x,y)\} = aG(k_x,k_y) + bH(k_x,k_y) \]

Computing Transforms

Similarly,

\[ F\{e^{j2\pi k_0 x}\} = \delta(k_x - k_0) \]

\[ F\{\cos 2\pi k_0 x\} = \frac{1}{2}\left(\delta(k_x - k_0) + \delta(k_x + k_0)\right) \]

\[ F\{\sin 2\pi k_0 x\} = \frac{1}{2j}\left(\delta(k_x - k_0) - \delta(k_x + k_0)\right) \]

Duality

Note the similarity between these two transforms

\[ F\{e^{j2\pi a x}\} = \delta(k_x - a) \]

\[ F\{\delta(x - a)\} = e^{-j2\pi k_0 a} \]

These are specific cases of duality

\[ F\{G(x)\} = g(-k_x) \]

Application of Duality

\[ F\{\text{sinc}(x)\} = \int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x} e^{-j2\pi k_0 x} dx = ?? \]

Recall that \( F\{\Pi(x)\} = \text{sinc}(k_x) \).

Therefore from duality, \( F\{\text{sinc}(x)\} = \Pi(-k_x) = \Pi(k_x) \)
2D Fourier Transform

Fourier Transform
\[ G(k_x, k_y) = \iint g(x, y) e^{-j2\pi(k_x x + k_y y)} dx dy \]

Inverse Fourier Transform
\[ g(x, y) = \iint G(k_x, k_y) e^{j2\pi(k_x x + k_y y)} dk_x dk_y \]

Plane Waves
\[ e^{j2\pi(k_x x + k_y y)} = \cos(2\pi(k_x x + k_y y)) + j \sin(2\pi(k_x x + k_y y)) \]

Plane Waves
\[
\begin{align*}
\Delta ABC &= \Delta ABDC \\
AC &= \frac{AB}{BC} = \frac{BD}{AD} \\
BD &= \frac{AB}{AC} \cdot \frac{1}{\cos \theta} = \frac{1}{\sqrt{k_x^2 + k_y^2}}
\end{align*}
\]

\[ \theta = \arctan \left( \frac{k_y}{k_x} \right) \]
Center of K-space

\[ G(0) = \lim_{\substack{\text{\(x\)} \to 0, \text{\(y\)} \to 0}} \int_\infty^{-\infty} g(x)e^{-j2\pi k x} dx \]

\[ = \int g(x)dx \]

\[ G(0) = \int_\infty^{-\infty} \int_\infty^{-\infty} g(x,y)e^{-j2\pi (k_x x + k_y y)} dx dy \]

\[ = \int g(x)dx \int_\infty^{-\infty} \]

Center of k-space is the area under the curve. Proportional to the mean value of the function.

Separable Functions

\( g(x,y) \) is said to be a separable function if it can be written as \( g(x,y) = g_x(x)g_y(y) \)

The Fourier Transform is then separable as well.

\[ G(k_x, k_y) = \int_\infty^{-\infty} \int_\infty^{-\infty} g(x,y)e^{-j2\pi (k_x x + k_y y)} dx dy \]

\[ = \int g_x(x)e^{-j2\pi k_x x} dx \int g_y(y)e^{-j2\pi k_y y} dy \]

\[ = G_x(k_x)G_y(k_y) \]

Example

\( g(x,y) = \Pi(x)\Pi(y) \)

\( G(k_x, k_y) = \text{sinc}(k_x)\text{sinc}(k_y) \)

Example (sinc/rect)

\( g(x,y) = \Pi(x)\Pi(y) \)

\( G(k_x, k_y) = \text{sinc}(k_x)\text{sinc}(k_y) \)

Examples

Is this function separable?
What is its Fourier Transform?

\[ g(x,y) = \exp(-j2\pi(8x + 9y))\sin(28\pi x) \]

PollEv.com/be280a
Examples

\[ g(x, y) = \delta(x, y) = \delta(x)\delta(y) \]
\[ G(k_x, k_y) = 1 \]

\[ g(x, y) = \delta(x) \]
\[ G(k_x, k_y) = \delta(k_y) \]

\[ g(x, y) = \cos(2\pi(ax - by)) \]
\[ G(k_x, k_y) = \frac{1}{2} \delta(k_x - a)\delta(k_y + b) + \frac{1}{2} \delta(k_x + a)\delta(k_y - b) \]

Examples

\[ g(x, y) = 1 + e^{-2\pi ax} \]
\[ G(k_x, k_y) = \delta(k_x) + \delta(k_x + a)\delta(k_y) \]

\[ g(x, y) = 1 + e^{-2\pi ay} \]
\[ G(k_x, k_y) = \delta(k_x) + \delta(k_x - a)\delta(k_y) \]

Scaling Theorem

\[ F\{g(ax)\} = \frac{1}{|a|} G\left(\frac{k_x}{a}\right) \]
\[ F\{g(ax, by)\} = \frac{1}{|ab|} G\left(\frac{k_x}{a}, \frac{k_y}{b}\right) \]
Modulation Transfer Function (MTF) or Frequency Response
Convolution/Modulation Theorem

\[
F\{g(x) * h(x)\} = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} g(u) * h(x-u) du \right] e^{-j2\pi kx} dx \\
= \int_{-\infty}^{\infty} g(u) \int_{-\infty}^{\infty} h(x-u) e^{-j2\pi kx} dx du \\
= \int_{-\infty}^{\infty} g(u) H(k_x) e^{-j2\pi kx} du \\
= G(k_x) H(k_x)
\]

Convolution in the spatial domain transforms into multiplication in the frequency domain. Dual is modulation

\[
F\{g(x) h(x)\} = G(k_x) * H(k_x)
\]
**Convolution/Multiplication**

Now consider an arbitrary input $h(x)$.

Recall that we can express $h(x)$ as the integral of weighted complex exponentials.

$$h(x) = \int_{-\infty}^{\infty} H(k) e^{j2\pi k x} dk$$

Each of these exponentials is weighted by $G(k)$ so that the response may be written as

$$z(x) = \int_{-\infty}^{\infty} G(k) H(k) e^{j2\pi k x} dk$$

**Eigenfunctions**

The fundamental nature of the convolution theorem may be better understood by observing that the complex exponentials are eigenfunctions of the convolution operator.

$$e^{j2\pi k x} \rightarrow \mathcal{F}(e^{j2\pi k x}) = \int_{-\infty}^{\infty} g(u) e^{j2\pi k (x-u)} du = G(k)e^{j2\pi k x}$$

The response of a linear shift invariant system to a complex exponential is simply the exponential multiplied by the FT of the system’s impulse response.

**Application of Convolution Thm.**

$$\Lambda(x) = \begin{cases} 1 - |x| & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$$

$$F(\Lambda(x)) = \int_{-1}^{1} (1-|k|) e^{-j2\pi k x} dk = ??$$

**Application of Convolution Thm.**

$$\Lambda(x) = \Pi(x) * \Pi(x)$$

$$F(\Lambda(x)) = \text{sinc}^2(k)$$
**Convolution Example**

System MTF = Product of MTFs of Components

**Response of an Imaging System**

Useful Approximation

\[
\text{FWHM}_{\text{System}} = \sqrt{\text{FWHM}_1^2 + \text{FWHM}_2^2 + \cdots + \text{FWHM}_N^2}
\]

**Example**

\[
\begin{align*}
\text{FWHM}_1 &= 1 \text{mm} \\
\text{FWHM}_2 &= 2 \text{mm} \\
\text{FWHM}_{\text{System}} &= \sqrt{5} = 2.24 \text{mm}
\end{align*}
\]
The intrinsic resolution of a gamma camera is 5 mm. The collimator resolution is 10 mm. The overall system resolution is ____ mm.

A. 15
B. 11.2
C. 7.5
D. 5.0
E. 0.5

Example

Amplitude Modulation (e.g. AM Radio)

\[ g(t) \rightarrow 2g(t) \cos(2\pi f_0 t) \]

\[ 2\cos(2\pi f_0 t) \]

\[ G(f) \]

\[ G(f-f_0) + G(f+f_0) \]

Modulation

\[ F\left[ g(x)e^{j2\pi f_0 x} \right] = G(k_x) \ast \delta(k_x - k_0) = G(k_x - k_0) \]

\[ F\left[ g(x)\cos(2\pi f_0 x) \right] = \frac{1}{2} G(k_x - k_0) + \frac{1}{2} G(k_x + k_0) \]

\[ F\left[ g(x)\sin(2\pi f_0 x) \right] = \frac{1}{2j} G(k_x - k_0) - \frac{1}{2j} G(k_x + k_0) \]
Shift Theorem

\[ F\{g(x-a)\} = F\{g(x)\ast \delta(x-a)\} = G(k_e)e^{-j2\pi ak}\]

\[ F\{g(x-a,y-b)\} = F\{g(x,y)\ast \delta(x-a,y-b)\} = G(k_x,k_y)e^{-j2\pi (ak_x+bk_y)}\]

Shifting the function doesn’t change its spectral content, so the magnitude of the transform is unchanged. Each frequency component is shifted by the amount \(a\). This corresponds to a relative phase shift of

\[-2\pi a/\text{(spatial period)} = -2\pi ak_x\]

For example, consider \(\exp(j2\pi k_x x)\). Shifting this by \(a\) yields \(\exp(j2\pi k_x(x-a)) = \exp(j2\pi k_x x)\exp(-j2\pi ak_x)\)

Summary of Basic Properties

**Linearity**

\[ F\{ag(x,y) + bh(x,y)\} = aG(k_x,k_y) + bH(k_x,k_y) \]

**Scaling**

\[ F\{g(ax,by)\} = \frac{1}{ab}G\left(\frac{k_x}{a},\frac{k_y}{b}\right) \]

**Duality**

\[ F\{g(k_x)\} = h(x) \]

**Shift**

\[ F\{g(x-a,y-b)\} = G(k_x,k_y)e^{-j2\pi (ak_x+bk_y)} \]

**Convolution**

\[ F\{g(x)\ast h(x)\} = G(k_x,k_y)H(k_x,k_y) \]

**Multiplication**

\[ F\{g(x,y)h(x,y)\} = G(k_x,k_y)H(k_x,k_y) \]

**Modulation**

\[ F\{g(x,y)e^{j2\pi (ax+by)}\} = G(k_x-k_a,k_y-k_b) \]