Convolution

\[ g[m] = g[0] \delta[m] + g[1] \delta[m-1] + g[2] \delta[m-2] \]

\[ h'[m',k] = L[\delta[m-k]] = h'[m'-k] \]

\[ y[m'] = L[g[m]] \]

\[ = L[g[0]\delta[m]] + g[1]\delta[m-1] + g[2]\delta[m-2] \]

\[ = h[0]L[\delta[m]] + L[h[1]\delta[m-1]] + L[h[2]\delta[m-2]] \]

\[ = g[0]L[\delta[m]] + g[1]L[\delta[m-1]] + g[2]L[\delta[m-2]] \]

\[ = \sum_{k=0}^{2} g[k]h[m'-k] \]

1D Convolution

\[ I(x) = \int_{-\infty}^{\infty} g(\xi)h(x-\xi)d\xi \]

\[ y(x) = g(x) \ast h(x) \]

Useful fact:

\[ g(x) \ast \delta(x-\Delta) = \int_{-\infty}^{\infty} g(\xi)\delta(x-\Delta-\xi)d\xi = g(x-\Delta) \]
1D Convolution Examples

\[
\begin{align*}
&\text{input:} \quad x = \frac{-3}{4} \quad \frac{1}{2} \\
&\text{convolved with:} \quad x = \frac{1}{2} \quad \frac{1}{2} \\
&\text{output:} \quad x = ?
\end{align*}
\]

\[
\begin{align*}
&\text{input:} \quad x = \frac{-1}{2} \quad \frac{1}{2} \\
&\text{convolved with:} \quad x = \frac{1}{2} \quad \frac{1}{2} \\
&\text{output:} \quad x = ?
\end{align*}
\]

2D Convolution

For a space invariant linear system, the superposition integral becomes a convolution integral.

\[
I(x_2, y_2) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi, \eta) h(x_2-x, y_2-y) d\xi d\eta
\]

\[
= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(\xi, \eta) h(x_2-\xi, y_2-\eta) d\xi d\eta
\]

\[
= g(x_2, y_2) * * h(x_2, y_2)
\]

where ** denotes 2D convolution. This will sometimes be abbreviated as *, e.g. \(I(x_2, y_2) = g(x_2, y_2) * h(x_2, y_2)\).

2D Convolution Example

\[
g(x) = \delta(x+1/2, y) + \delta(x, y) \quad h(x) = \text{rect}(x, y)
\]

\[
I(x, y) = g(x) ** h(x, y)
\]

\[
\begin{align*}
&\text{input:} \quad g(x) = \frac{-1}{2} \quad \frac{1}{2} \\
&\text{convolved with:} \quad h(x) = \frac{-1}{2} \quad \frac{1}{2} \\
&\text{output:} \quad I(x, y) = \frac{-1}{2} \quad \frac{1}{2}
\end{align*}
\]
For off-center pinhole object, the shifted source image can be written as
\[ s\left(\frac{x - Mx_0}{m}\right) = s\left(\frac{x}{m}\right) * \frac{1}{M} \delta\left(\frac{x - Mx_0}{M}\right) = s\left(\frac{x}{m}\right) * \delta\left(\frac{x}{M}ight) \]

For the general 2D case, we convolve the magnified object with the impulse response
\[ I(x, y) = t\left(\frac{x}{M}, \frac{y}{M}\right) * * \frac{1}{m^2} s\left(\frac{x}{m}, \frac{y}{m}\right) \]

Note: we have ignored obliquity factors etc.
X-Ray Imaging

\[ m = 1; M = 2 \]

\[
\frac{1}{m} \left( f(\frac{x}{m}) \right) = \text{rect}(x/10) \ast \text{rect}(x/20)
\]

= ???

Summary

1. The response to a linear system can be characterized by a spatially varying impulse response and the application of the superposition integral.
2. A shift invariant linear system can be characterized by its impulse response and the application of a convolution integral.