Bioengineering 280A  
Principles of Biomedical Imaging  
Fall Quarter 2004  
Lecture 5  
Sampling

Topics
1. Overview of Sampling
2. 1D Sampling
3. 2D Sampling
4. Aliasing

 Analog vs. Digital
The Analog World:  
Continuous time/space, continuous valued signals or images, e.g. vinyl records, photographs, x-ray films.

The Digital World:  
Discrete time/space, discrete-valued signals or images, e.g. CD-Roms, DVDs, digital photos, digital x-rays, CT, MRI, ultrasound.
The Process of Sampling

\[ g(x) \]

\[ g[n] = g(n \Delta x) \]

Questions

How finely do we need to sample?

What happens if we don’t sample finely enough?

Can we reconstruct the original signal or image from its samples?

Sampling in the Time Domain
Comb Function

\[ \text{comb}(x) = \sum_{n=-\infty}^{\infty} \delta(x - n) \]

Other names: Impulse train, bed of nails, shah function.

Scaled Comb Function

\[ \text{comb}\left(\frac{x}{\Delta x}\right) = \sum_{n=-\infty}^{\infty} \frac{x - n\Delta x}{\Delta x} \]

\[ = \Delta x \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x) \]

1D spatial sampling

\[ g_{\xi}(x) = g(x) \frac{1}{\Delta x} \text{comb}\left(\frac{x}{\Delta x}\right) \]

\[ = g(x) \sum_{n=-\infty}^{\infty} \delta(x - n\Delta x) \]

\[ = \sum_{n=-\infty}^{\infty} g(n\Delta x) \delta(x - n\Delta x) \]

Recall the sifting property \( \int g(x) \delta(x - a) = g(a) \)

But we can also write \( \int g(x) \delta(x - a) = g(a) \int \delta(x - a) = g(a) \)

So, \( g(x) \delta(x - a) = g(a) \delta(x - a) \)
1D spatial sampling

$g(x)$

$\text{comb}(x/\Delta x)/\Delta x$

$\Delta x$

$g(x)$

Fourier Transform of $\text{comb}(x)$

$F[\text{comb}(x)] = \text{comb}(k_x)$

$= \sum_{n=-\infty}^{\infty} \delta(k_x - n)$

$F\left[ \frac{1}{\Delta x} \text{comb}(\frac{x}{\Delta x}) \right] = \frac{1}{\Delta x} \Delta x \text{comb}(k_x \Delta x)$

$= \sum_{n=-\infty}^{\infty} \delta(k_x \Delta x - n)$

$= \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} \delta(k_x - \frac{n}{\Delta x})$

Fourier Transform of $\text{comb}(x/\Delta x)$

$\text{comb}(x/\Delta x)/\Delta x$

$\Delta x$

$F$

$1/\Delta x$

$\text{comb}(k_x \Delta x)$

$1/\Delta x$

$k_x$
Fourier Transform of \(g_S(x)\)

\[
F[g_S(x)] = F\left[\frac{1}{\Delta x} \text{comb}\left(\frac{x}{\Delta x}\right)\right] \\
= G(k_x) * F\left[\frac{1}{\Delta x} \text{comb}\left(\frac{x}{\Delta x}\right)\right] \\
= G(k_x) * \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} \delta(k_x - \frac{n}{\Delta x}) \\
= \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} G(k_x) * \delta(k_x - \frac{n}{\Delta x}) \\
= \frac{1}{\Delta x} \sum_{n=-\infty}^{\infty} G\left(k_x - \frac{n}{\Delta x}\right)
\]

Nyquist Condition

To avoid overlap, we require that \(\Delta x > 2B\) or \(K_S > 2B\) where \(K_S = 1/\Delta x\) is the sampling frequency.
Reconstruction from Samples

If the Nyquist condition is met, then
\[ \hat{G}_s(k_x) = \frac{1}{K_s} G_s(k_x) \text{rect}(k_x / K_s) = G(k_x) \]

And the signal can be reconstructed by convolving the sample with a sinc function
\[ \hat{g}_s(x) = g_s(x) * \text{sinc}(K_s x) \]
\[ = \sum_{n=-\infty}^{\infty} g(n \Delta X) \delta(x - n \Delta X) * \text{sinc}(K_s x) \]
\[ = \sum_{n=-\infty}^{\infty} g(n \Delta X) \text{sinc}(K_s(x - n \Delta X)) \]

Example Cosine Reconstruction

\[ \cos(2\pi k_0 x) \]
Cosine Example with $K_z = 2k_0$

Example with $K_z = 4k_0$

Example with $K_z = 8k_0$
Example Sine Reconstruction

\[ 2\sin(2\pi k_0 x) \]

\( K_s \geq 2k_0 \)

Aliasing

\[ G(k_s) \]

Aliasing occurs when the Nyquist condition is not satisfied. This occurs for \( K_s \neq 2B \)
Aliasing Example

\[ \cos(2\pi k_0 x) \]

\[ k_0 \quad k_0 \]

\[ k_0 \quad k_0 \]

\[ K_S \]

\[ 2k_0 > K_S > k_0 \]

\[ K_S = k_0 \]
Practical Considerations

Why sample higher than the Nyquist frequency?

- true sinc interpolation is not practical since the sinc function goes from -infinity to infinity
- the requirements on the low-pass filter are reduced.

\[ k_0 - k \to k_0 \]

Hard

Easier

Fourier Sampling

Instead of sampling the signal, we sample its Fourier Transform

\[ G_k(kx) = \sum_{n=-\infty}^{\infty} G(n\Delta k - n\Delta k_x) \]

\[ G_k(kx) = G(kx) \frac{1}{\Delta k} \text{comb}(kx/\Delta k) \]

\[ G_k(kx) = G(kx) \sum_{n=-\infty}^{\infty} \delta(n\Delta k - n\Delta k_x) \]

\[ G_k(kx) = \sum_{n=-\infty}^{\infty} G(n\Delta k) \delta(kx - n\Delta k) \]
Fourier Sampling -- Inverse Transform

\[
x \cdot \frac{1}{\Delta k_x} = \frac{1}{\Delta k_x}
\]

Nyquist Condition

To avoid overlap, \(1/\Delta k_x > \text{FOV}\), or equivalently, \(\Delta k_x < 1/\text{FOV}\)
Aliasing occurs when \( 1/\Delta k_x < \text{FOV} \)

**2D Comb Function**

\[
\text{comb}(x, y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m, y - n)
= \sum_{m=-\infty}^{\infty} \delta(x - m) \delta(y - n)
= \text{comb}(x) \text{comb}(y)
\]
Scaled 2D Comb Function

\[
\text{comb}(x/\Delta x, y/\Delta y) = \text{comb}(x/\Delta x)\text{comb}(y/\Delta y)
\]

\[
= \Delta x\Delta y \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x-m\Delta x)\delta(y-n\Delta y)
\]

2D k-space sampling

\[
G_z(k_x,k_y) = \frac{1}{\Delta k_x\Delta k_y} \text{comb}\left(\frac{k_x}{\Delta k_x}, \frac{k_y}{\Delta k_y}\right)
\]

\[
= G(k_x,k_y) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(k_x-m\Delta k_x, k_y-n\Delta k_y)
\]

\[
= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} G(m\Delta k_x, n\Delta k_y)\delta(k_x-m\Delta k_x, k_y-n\Delta k_y)
\]
2D k-space sampling

\[ g(x, y) = F \left[ F^{-1} \left[ g(k_x, k_y) \right] \right] \]

Nyquist Conditions

\[ \frac{1}{\Delta k_x} > \text{FOV}_x \]
\[ \frac{1}{\Delta k_y} > \text{FOV}_y \]

Aliasing