Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2005
MRI Lecture 1

Topics

• The concept of spin
• Precession of magnetic spin
• Relaxation
• Bloch Equation

Spin

• Intrinsic angular momentum of elementary particles -- electrons, protons, neutrons.
• Spin is quantized. Key concept in Quantum Mechanics.
The History of Spin

- 1921 Stern and Gerlach observed quantization of magnetic moments of silver atoms.
- 1925 Uhlenbeck and Goudsmit introduce the concept of spin for electrons.
- 1933 Stern and Gerlach measure the effect of nuclear spin.
- 1937 Rabi predicts and observes nuclear magnetic resonance.

Classical Magnetic Moment

\[ \vec{\mu} = I \hat{\alpha} \]

Energy in a Magnetic Field

\[ E = -\vec{\mu} \cdot \vec{B} = -\mu z B \]
Stern-Gerlach Experiment

Force in a Field Gradient

\[ F = -\nabla E = \mu_0 \frac{\partial B_z}{\partial z} \]

Deflected up

Deflected down

Increasing vertical B-field.

Stern-Gerlach Experiment
The key finding of the Stern-Gerlach experiment is that the magnetic moment is quantized. That is, it can only take on discrete values. In the experiment, the finding was that the component of magnetization along the direction of the applied field was quantized:

\[ \mu_z = \pm \mu_0 \text{ OR } -\mu_0 \]

A charged sphere spinning about its axis has angular momentum and a magnetic moment. This is a classical analogy that is useful for understanding quantum spin, but remember that it is only an analogy!

Relation: \( \mathbf{\mu} = \gamma \mathbf{S} \) where \( \gamma \) is the gyromagnetic ratio and \( \mathbf{S} \) is the spin angular momentum.

Because the magnetic moment is quantized, so is the angular momentum. In particular, the z-component of the angular momentum Is quantized as follows:

\[ L_z = m_l \hbar \quad m_l \in \{-s, -(s-1), \ldots, s\} \]

\( s \) is an integer or half integer.
Nuclear Spin Rules

<table>
<thead>
<tr>
<th>Number of Protons</th>
<th>Number of Neutrons</th>
<th>Spin</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Even</td>
<td>Even</td>
<td>0</td>
<td>$^{12}$C, $^{16}$O</td>
</tr>
<tr>
<td>Even</td>
<td>Odd</td>
<td>$j/2$</td>
<td>$^{2}$H</td>
</tr>
<tr>
<td>Odd</td>
<td>Even</td>
<td>$j/2$</td>
<td>$^{23}$Na, $^{31}$P</td>
</tr>
<tr>
<td>Odd</td>
<td>Odd</td>
<td>$j$</td>
<td>$^4$H</td>
</tr>
</tbody>
</table>

Hydrogen Proton

Spin 1/2

\[ S_z = \begin{cases} +\hbar/2 \\ -\hbar/2 \end{cases} \]
\[ \mu_z = \begin{cases} +\gamma\hbar/2 \\ -\gamma\hbar/2 \end{cases} \]

Boltzmann Distribution

Number Spins Up \( \frac{\text{Number Spins Down}}{\text{Number Spins Down}} = \exp(-\Delta E/kT) \)

Ratio = 0.999990 at 1.5T !!!
Corresponds to an excess of about 10 up spins per million
Equilibrium Magnetization

\[ M = N \mu_z = N \left( \frac{\mu_z}{N} - \mu_z \right) \]
\[ = N \mu_z \left( e^{\frac{\mu_z B}{kT}} - e^{-\frac{\mu_z B}{kT}} \right) \]
\[ = N \gamma^2 \hbar^2 B / (4kT) \]

N = number of nuclear spins per unit volume
Magnetization is proportional to applied field.

Bigger is better

3T Human imager at UCSD
7T Human imager at U. Minn.
7T Rodent Imager at UCSD
9.4T Human imager at UIC

Gyromagnetic Ratios

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>Spin</th>
<th>Magnetic Moment</th>
<th>( \gamma (2\pi) ) (MHz/Tesla)</th>
<th>Abundance</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^1\text{H})</td>
<td>1/2</td>
<td>2.793</td>
<td>42.58</td>
<td>88 M</td>
</tr>
<tr>
<td>(^23\text{Na})</td>
<td>3/2</td>
<td>2.216</td>
<td>11.27</td>
<td>80 mM</td>
</tr>
<tr>
<td>(^31\text{P})</td>
<td>1/2</td>
<td>1.131</td>
<td>17.25</td>
<td>75 mM</td>
</tr>
</tbody>
</table>
Torque

For a non-spinning magnetic moment, the torque will try to align the moment with magnetic field (e.g. compass needle)

\[ \mathbf{N} = \mu \times \mathbf{B} \]

Precession

\[ \frac{dS}{dt} = N \]

Change in Angular momentum

Relation between magnetic moment and angular momentum

\[ \frac{d\mu}{dt} = \mu \times \gamma \mathbf{B} \]

Precession

Analogous to motion of a gyroscope

Precesses at an angular frequency of

\[ \omega = \gamma B \]

This is known as the Larmor frequency.
Larmor Frequency

\[ \omega = \gamma B \quad \text{Angular frequency in rad/sec} \]

\[ f = \frac{\gamma B}{2\pi} \quad \text{Frequency in cycles/sec or Hertz, Abbreviated Hz} \]

For a 1.5 T system, the Larmor frequency is 63.86 MHz which is 63.86 million cycles per second. For comparison, KPBS-FM transmits at 89.5 MHz.

Note that the earth’s magnetic field is about 50 µT, so that a 1.5T system is about 30,000 times stronger.

Notation and Units

1 Tesla = 10,000 Gauss
Earth’s field is about 0.5 Gauss
0.5 Gauss = 0.5x10⁻⁴ T = 50 µT

\[ \gamma = 26752 \text{ radians/second/Gauss} \]

\[ \varphi = \frac{\gamma}{2\pi} = 4258 \text{ Hz/Gauss} = 42.58 \text{ MHz/Tesla} \]

Magnetization Vector

Vector sum of the magnetic moments over a volume.
For a sample at equilibrium in a magnetic field, the transverse components of the moments cancel out, so that there is only a longitudinal component.
Equation of motion is the same form as for individual moments.

\[ M = \frac{1}{V} \sum_{\text{protons}} \mu_i \]

\[ \frac{dM}{dt} = \gamma M \times B \]

http://www.easymeasure.co.uk/principlemri.aspx
RF Excitation

At equilibrium, net magnetization is parallel to the main magnetic field. How do we tip the magnetization away from equilibrium?

$B_1$, radiofrequency field tuned to Larmor frequency and applied in transverse ($xy$) plane induces nutation (at Larmor frequency) of magnetization vector as it tips away from the z-axis.

- lab frame of reference
a) Laboratory frame behavior of \( \mathbf{M} \)

Images & caption: Nakamura, Fig. 3.3

b) Rotating frame behavior of \( \mathbf{M} \)

RF Excitation

From Buxton 2002

Free Induction Decay (FID)

http://www.easymeasure.co.uk/principlesmri.aspx
Relaxation

An excitation pulse rotates the magnetization vector away from its equilibrium state (purely longitudinal). The resulting vector has both longitudinal $M_z$ and transverse $M_{xy}$ components.

Due to thermal interactions, the magnetization will return to its equilibrium state with characteristic time constants.

- $T_1$ spin-lattice time constant, return to equilibrium of $M_z$
- $T_2$ spin-spin time constant, return to equilibrium of $M_{xy}$

Longitudinal Relaxation

$$\frac{dM_z}{dt} = -\frac{M_z - M_0}{T_1}$$

After a 90 degree pulse $M_z(t) = M_0(1 - e^{-t/T_1})$

Due to exchange of energy between nuclei and the lattice (thermal vibrations). Process continues until thermal equilibrium as determined by Boltzmann statistics is obtained.

The energy $\Delta E$ required for transitions between down to up spins, increases with field strength, so that $T_1$ increases with $B$. 
T1 Values

Transverse Relaxation

\[ \frac{dM_{xy}}{dt} = \frac{M_{xy}}{T_2} \]

Each spin’s local field is affected by the z-component of the field due to other spins. Thus, the Larmor frequency of each spin will be slightly different. This leads to a dephasing of the transverse magnetization, which is characterized by an exponential decay.

\[ T_2 \] is largely independent of field. \( T_2 \) is short for low frequency fluctuations, such as those associated with slowly tumbling macromolecules.

T2 Relaxation

After a 90 degree excitation

\[ M_{xy}(t) = M_0 e^{-t/T_2} \]
**T2 Relaxation**

![Diagram showing T2 Relaxation]

**T2 Values**

<table>
<thead>
<tr>
<th>Tissue</th>
<th>T₂ (ms)</th>
</tr>
</thead>
<tbody>
<tr>
<td>gray matter</td>
<td>100</td>
</tr>
<tr>
<td>white matter</td>
<td>92</td>
</tr>
<tr>
<td>muscle</td>
<td>47</td>
</tr>
<tr>
<td>fat</td>
<td>85</td>
</tr>
<tr>
<td>kidney</td>
<td>58</td>
</tr>
<tr>
<td>liver</td>
<td>43</td>
</tr>
<tr>
<td>CSF</td>
<td>4000</td>
</tr>
</tbody>
</table>

Table: adapted from Nishimura, Table 4.2

Solids exhibit very short T₂ relaxation times because there are many low frequency interactions between the immobile spins.

On the other hand, liquids show relatively long T₂ values, because the spins are highly mobile and net fields average out.

**Example**

![MRI images showing T₁, T₂, and density-weighted images]

Questions: How can one achieve T₂ weighting? What are the relative T₂'s of the various tissues?
Bloch Equation

\[
\frac{d\mathbf{M}}{dt} = \mathbf{M} \times \gamma \mathbf{B} - \frac{M_x \mathbf{i} + M_y \mathbf{j}}{T_2} - \frac{(M_z - M_0) \mathbf{k}}{T_1}
\]

- Precession
- Transverse Relaxation
- Longitudinal Relaxation

k, j, k are unit vectors in the x,y,z directions.

Free precession about static field

\[
\frac{d\mathbf{M}}{dt} = \mathbf{M} \times \gamma \mathbf{B} = \gamma \mathbf{B} \times \mathbf{M} \quad \text{or} \quad \begin{bmatrix} i & j & k \\ M_x & M_y & M_z \\ B_x & B_y & B_z \end{bmatrix} = \gamma \begin{bmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{bmatrix} \begin{bmatrix} M_x \\ M_y \\ M_z \end{bmatrix}
\]

Free precession about static field

\[
\begin{bmatrix} \frac{dM_x}{dt} \\ \frac{dM_y}{dt} \\ \frac{dM_z}{dt} \end{bmatrix} = \gamma \begin{bmatrix} B_x M_z - B_z M_x \\ B_y M_z - B_z M_y \\ B_z M_y - B_y M_z \end{bmatrix}
\]
Precession

\[
\begin{bmatrix}
\frac{dM_x}{dt} \\
\frac{dM_y}{dt} \\
\frac{dM_z}{dt}
\end{bmatrix}
= \begin{bmatrix}
0 & B_0 & 0 \\
-B_0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
\]

Useful to define \( M = M_x + jM_y \)

\[
\frac{dM}{dt} = d/dt (M_x + iM_y) = -jB_0 M
\]

Solution is a time-varying phasor

\[ M(t) = M(0)e^{-j\omega t} = M(0)e^{-j\omega_0 t} \]

Question: which way does this rotate with time?

Matrix Form with \( B = B_0 \)

\[
\begin{bmatrix}
\frac{dM_x}{dt} \\
\frac{dM_y}{dt} \\
\frac{dM_z}{dt}
\end{bmatrix}
= \begin{bmatrix}
-1/T_1 & yB_0 & 0 \\
-B_0 & 1/T_1 & 0 \\
0 & 0 & -1/T_1
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
\]

Z-component solution

\[ M_z(t) = M_0 + (M_z(0) - M_0)e^{-t/T_1} \]

Saturation Recovery

If \( M_z(0) = 0 \) then \( M_z(t) = M_0(1 - e^{-t/T_1}) \)

Inversion Recovery

If \( M_z(0) = -M_0 \) then \( M_z(t) = M_0(1 - 2e^{-t/T_1}) \)
Transverse Component

\[ M = M_x + jM_y \]

\[ \frac{dM}{dt} \equiv \frac{d}{dt}(M_x + iM_y) = -j(\omega_0 + 1/T_\text{T})M \]

\[ M(t) = M(0)e^{-\omega_0 t}e^{-i/T_\text{T}} \]

Summary

1) Longitudinal component recovers exponentially.
2) Transverse component precesses and decays exponentially.

Fact: Can show that \( T_2 < T_1 \) in order for \( |M(t)| \leq M_0 \). Physically, the mechanisms that give rise to \( T_1 \) relaxation also contribute to transverse \( T_2 \) relaxation.