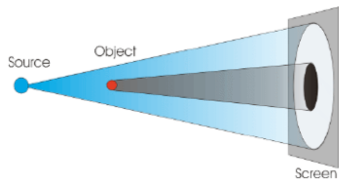
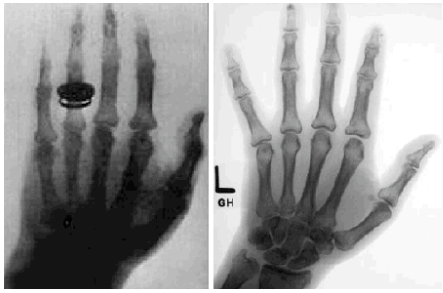
Bioengineering 280A
Principles of Biomedical Imaging

Fall Quarter 2007
X-Rays Lecture 2

Topics

- Review topics from last lecture
- Attenuation
- Contrast
- X-Ray Imaging Equation

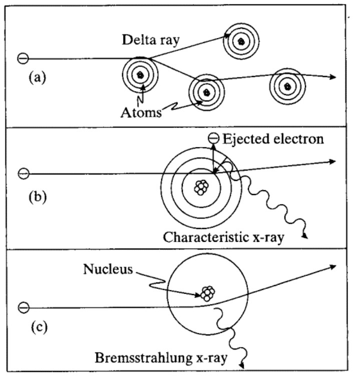




X-Ray Production

Collisional transfers

Radiative transfers



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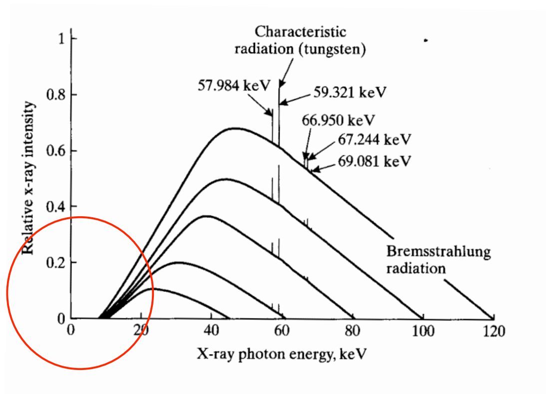
X-Ray Spectrum



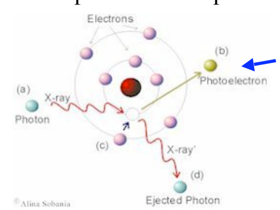
Lower energy photons are absorbed by anode, tube, and other filters

Prince and Links 2005

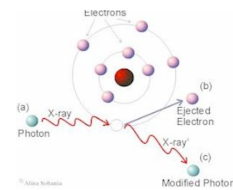
Interaction with Matter

Typical energy range for diagnostic x-rays is below 200 keV.

The two most important types of interaction are photoeletric absorption and Compton scattering.



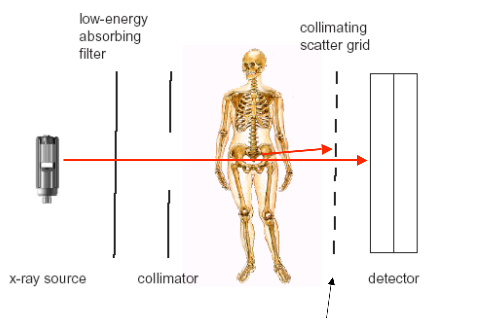
Photoelectric effect dominates at low x-ray energies and high atomic numbers. $E_{e^-} = \hbar\nu - E_B$



Compton scattering dominates at high x-ray energies and low atomic numbers, not much contrast

http://www.eee.ntu.ac.uk/research/vision/asobania

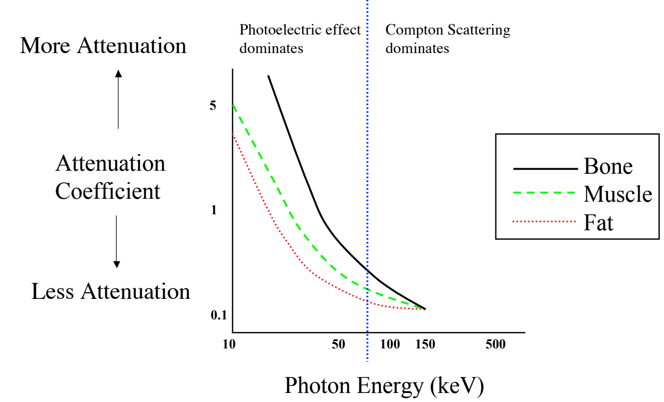
X-Ray Imaging Chain



Reduces effects of Compton scattering

Suetens 2002

Attenuation



Adapted from www.cis.rit.edu/class/simg215/xrays.ppt

Intensity

$I = E\phi$

Energy (E) — Photon flux rate (ϕ)

$$\phi = \frac{N}{A\Delta t}$$

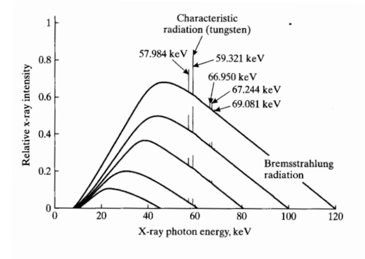
Number of photons (N)

Unit Area (A) — Unit Time (Δt)

Intensity

$$\phi = \int_0^\infty S(E')dE'$$

X-ray spectrum



$$I = \int_0^\infty S(E')E'dE'$$

Attenuation

$n = \mu N \Delta x$ photons lost per unit length

$\mu = \dfrac{n/N}{\Delta x}$ fraction of photons lost per unit length

$$\Delta N = -n \longrightarrow \frac{dN}{dx} = -\mu N \longrightarrow N(x) = N_0 e^{-\mu x}$$

For mono-energetic case, intensity is

$$I(\Delta x) = I_0 e^{-\mu \Delta x}$$

Attenuation

Inhomogeneous Slab

$$\frac{dN}{dx} = -\mu(x)N \longrightarrow N(x) = N_0 \exp\left(-\int_0^x \mu(x')dx'\right)$$

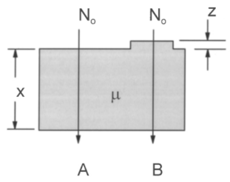
$$I(x) = I_0 \exp\left(-\int_0^x \mu(x')dx'\right)$$

Attenuation depends on energy, so also need to integrate over energies

$$I(x) = \int_0^\infty S_0(E')E' \exp\left(-\int_0^x \mu(x';E')dx'\right)dE'$$

Background intensity → $A = N_0 \exp(-\mu x)$

Object intensity → $B = N_0 \exp(-\mu(x+z))$



(A) X-ray Imaging

Bushberg et al 2001

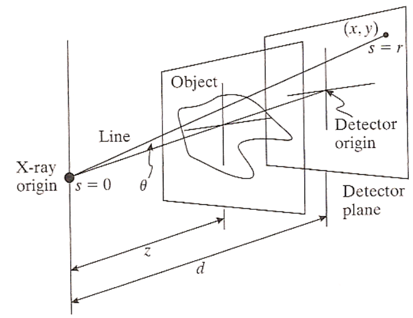
Contrast

$$C = \frac{B-A}{B+A} = \frac{N_0 \exp(-\mu(x+z)) - N_0 \exp(-\mu x)}{N_0 \exp(-\mu(x+z)) + N_0 \exp(-\mu x)}$$

Subject/Local Contrast

$$C_S = \frac{B-A}{A} = \frac{N_0 \exp(-\mu(x+z)) - N_0 \exp(-\mu x)}{N_0 \exp(-\mu x)} = \exp(-\mu z) - 1$$

X-Ray Imaging Geometry



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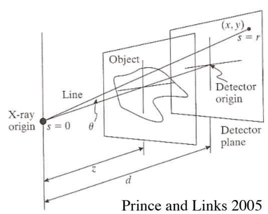
Inverse Square Law

Inverse Square Law

$$I_0 = \frac{I_S}{4\pi d^2}$$

$$I_d(x,y) = \frac{I_S}{4\pi r^2} \quad \text{where } r^2 = x^2 + y^2 + d^2$$

$$= \frac{I_0 d^2}{r^2} = I_0 \cos^2\theta$$



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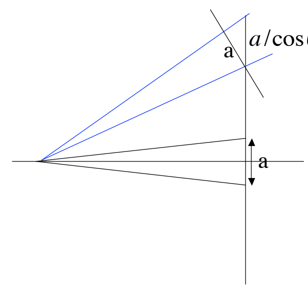
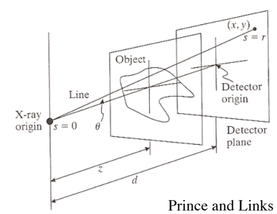
Obliquity Factor

Obliquity Factor

$I_d(x,y) = I_0 \cos\theta$





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X-Ray Imaging Geometry

Beam Divergence and Flat Panel

$I_r = I_0 \cos^3 \theta$

Example: Chest x-ray at 2 yards with 14x17 inch film.

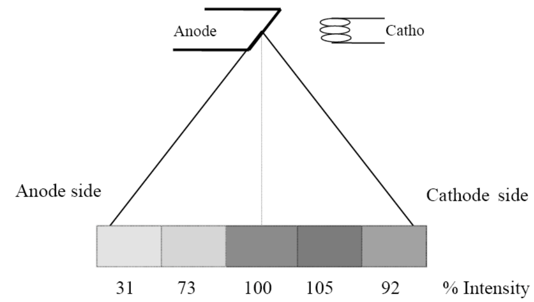
Question: What is the smallest ratio I_r/I_0 across the film?

$$r_d = \sqrt{7^2 + 8.5^2} = 11$$

$$\cos\theta = \frac{d}{\sqrt{r_d^2 + d^2}} = 0.989$$

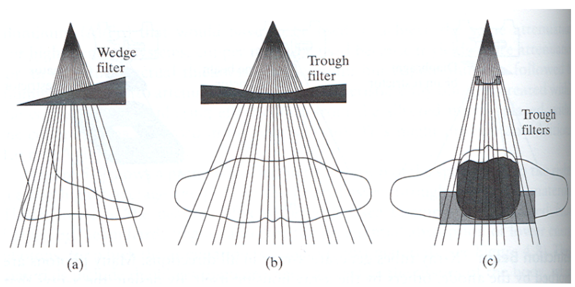
$$\frac{I_r}{I_0} = \cos^3\theta = 0.966$$

Anode Heel Effect



http://www.animalinsides.com/radphys/chapters/Lect2.pdf

Compensation Filters



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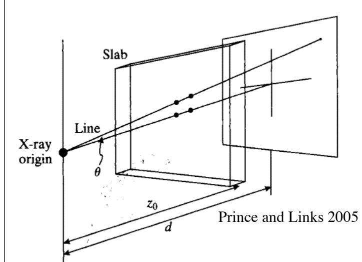
Path Length



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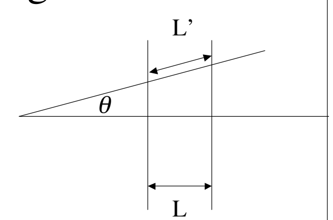
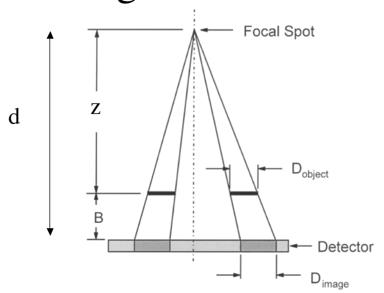


$$L' = L/\cos\theta$$

$$I_d(x,y) = I_0 \cos^3\theta \exp(-\mu L/\cos\theta)$$

Magnification of Object

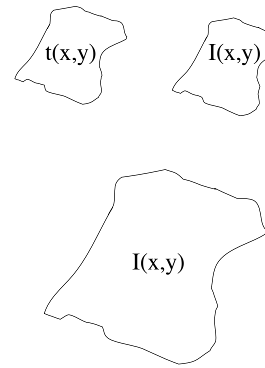


$$M(z) = \frac{d}{z}$$

$$= \frac{\text{Source to Image Distance (SID)}}{\text{Source to Object Distance (SOD)}}$$

Bushberg et al 2001

Magnification of Object



M = 1: I(x,y) = t(x,y)

M = 2: I(x,y) = t(x/2,y/2)

In general, I(x,y) = t(x/M(z),y/M(z))

X-Ray Imaging Equation

At z = d there is no magnification, so

$$I_d(x,y) = I_0 \cos^3\theta \cdot \exp\left(-\int_{L_{x,y}} \mu(s)ds \Big/ \cos\theta \right)$$

$$= I_0 \cos^3\theta \cdot t_d(x,y)$$

where $t_z(x,y)$ is the transmittivity of the object at distance z

In general, with magnification

$$I_d(x,y) = I_0 \cos^3\theta \cdot t_z(x/M(z), y/M(z))$$

Prince and Links 2005

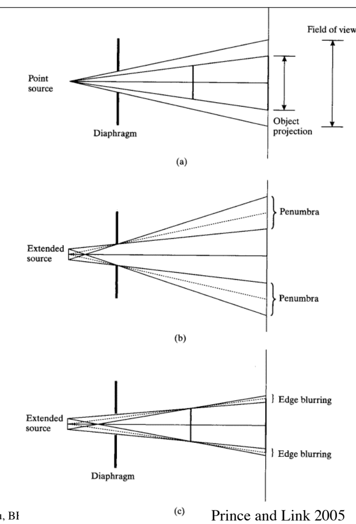
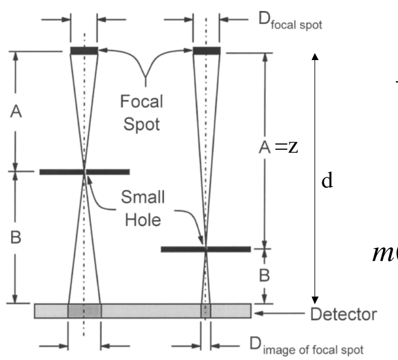


Figure 5.18
Effects of extended source. (a) Ideal field of view and object projection (with magnification). (b) Penumbra at edges of field of view due to extended source. (c) Blurred object edges due to extended source.

Prince and Link 2005

Source magnification



Bushberg et al 2001

$$\frac{D_{image}}{D_{focal}} = \frac{d-z}{z}$$

$$m(z) = -\frac{d-z}{z} = -\frac{B}{A}$$
$$= 1 - M(z)$$

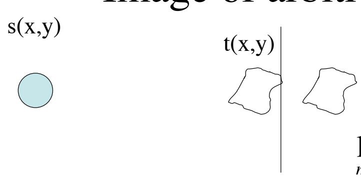
Image of a point object

$$I_d(x,y) = ks(x/m, y/m)$$

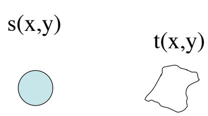
$$\int\int ks(x/m(z), y/m(z))dxdy = \text{ constant}$$
$$\Rightarrow k = \frac{1}{m^2(z)}$$

$$I_d(x,y) = \lim_{m \to 0} \frac{s(x/m, y/m)}{m^2}$$
$$= \delta(x,y)$$

Image of arbitrary object



$$\lim_{m \to 0} I_d(x,y) = t(x,y)$$

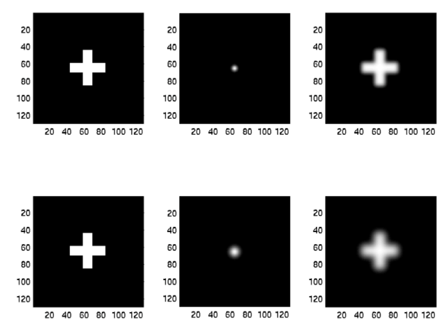


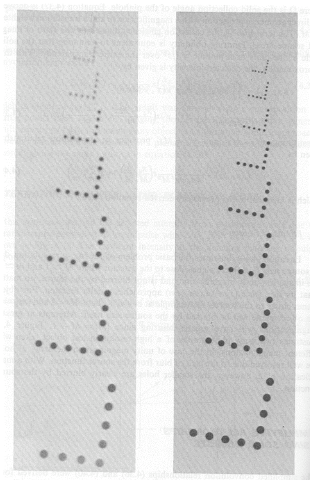
m=1

$$I_d(x,y) = ???$$

$$I_d(x,y) = \frac{\cos^3\theta}{4\pi d^2 m^2} s(x/m, y/m) ** t(x/M, y/M)$$

Convolution



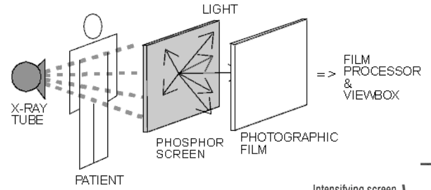


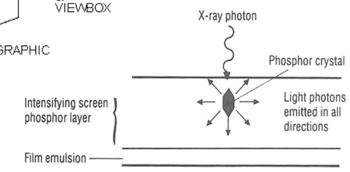
M=2
m=-1

M=1
m=0

Macovski 1983

Film-screen blurring



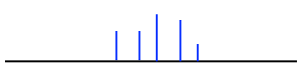


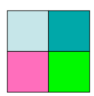
$$I_d(x,y) = \frac{\cos^3\theta}{4\pi d^2 m^2} s(x/m, y/m) ** t(x/M, y/M) ** h(x,y)$$

http://learntech.uwe.ac.uk/radiography/RScience/imaging_principles_d/diagimage11.htm
http://www.sunnybrook.utoronto.ca:8080/~selenium/xray.html#Film

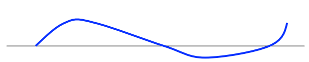
Signals and Images

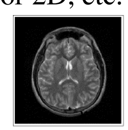
Discrete-time/space signal/image: continuous valued function with a discrete time/space index, denoted as *s[n]* for 1D, *s[m,n]* for 2D , etc.





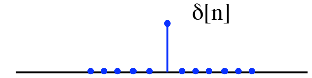
Continuous-time/space signal/image: continuous valued function with a continuous time/space index, denoted as *s(t)* or *s(x)* for 1D, *s(x,y)* for 2D, etc.

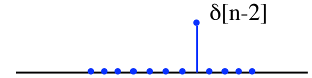




Kronecker Delta Function

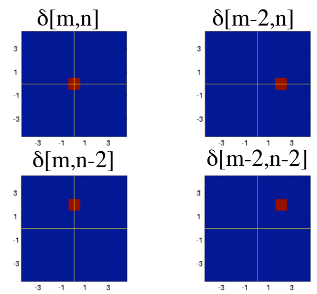
$$\delta[n] = \begin{cases} 1 & \text{for } n = 0 \\ 0 & \textit{otherwise} \end{cases}$$





Kronecker Delta Function

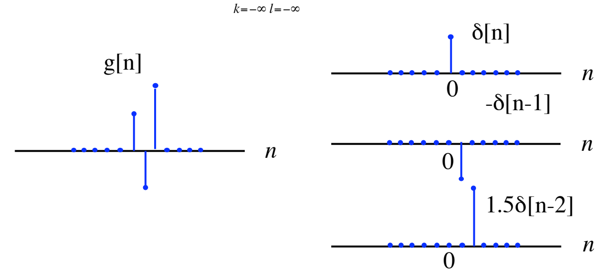
$$\delta[m,n] = \begin{cases} 1 & \text{for } m=0, n=0 \\ 0 & otherwise \end{cases}$$



Discrete Signal Expansion

$$g[n] = \sum_{k=-\infty}^{\infty} g[k]\delta[n-k]$$

$$g[m,n] = \sum_{k=-\infty}^{\infty}\sum_{l=-\infty}^{\infty} g[k,l]\delta[m-k,n-l]$$



2D Signal

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & d \end{bmatrix}$$

Image Decomposition

$$\begin{bmatrix} c & d \\ a & b \end{bmatrix} = c\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + a\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$g[m,n] = a\delta[m,n] + b\delta[m,n-1] + c\delta[m-1,n] + d\delta[m-1,n-1]$$

$$= \sum_{k=0}^{1}\sum_{l=0}^{1} g[k,l]\delta[m-k,n-l]$$

Dirac Delta Function

Notation:

$\delta(x)$ - 1D Dirac Delta Function

$\delta(x,y)$ or ${}^2\delta(x,y)$ - 2D Dirac Delta Function

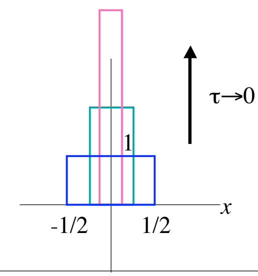
$\delta(x,y,z)$ or ${}^3\delta(x,y,z)$ - 3D Dirac Delta Function

$\delta(\vec{r})$ - N Dimensional Dirac Delta Function

1D Dirac Delta Function

$\delta(x) = 0$ when $x \neq 0$ and $\int_{-\infty}^{\infty} \delta(x)dx = 1$

Can interpret the integral as a limit of the integral of an ordinary function that is shrinking in width and growing in height, while maintaining a constant area. For example, we can use a shrinking rectangle function such that $\int_{-\infty}^{\infty} \delta(x)dx = \lim_{\tau \to 0} \int_{-\infty}^{\infty} \tau^{-1}\Pi(x/\tau)dx$.



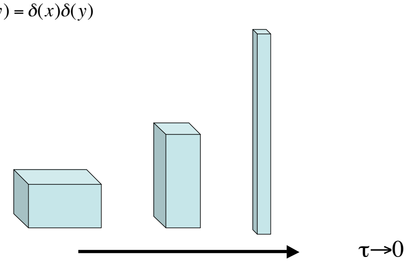
2D Dirac Delta Function

$\delta(x,y) = 0$ when $x^2 + y^2 \neq 0$ and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)dxdy = 1$

where we can consider the limit of the integral of an ordinary 2D function that is shrinking in width but increasing in height while maintaining constant area.

$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(x,y)dxdy = \lim_{\tau \to 0} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \tau^{-2}\Pi(x/\tau, y/\tau)dxdy.$

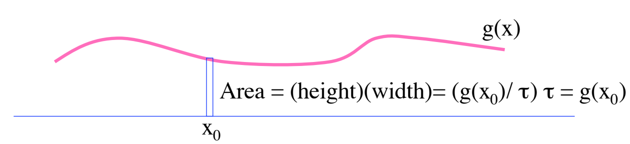
Useful fact: $\delta(x,y) = \delta(x)\delta(y)$



Generalized Functions

Dirac delta functions are not ordinary functions that are defined by their value at each point. Instead, they are generalized functions that are defined by what they do underneath an integral.

The most important property of the Dirac delta is the sifting property $\int_{-\infty}^{\infty} \delta(x - x_0)g(x)dx = g(x_0)$ where $g(x)$ is a smooth function. This sifting property can be understood by considering the limiting case $\lim_{\tau \to 0} \int_{-\infty}^{\infty} \tau^{-1}\Pi(x/\tau)g(x)dx = g(x_0)$

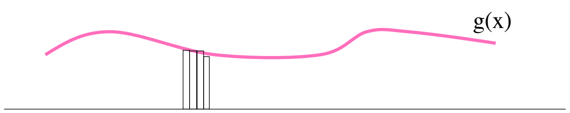


Representation of 1D Function

From the sifting property, we can write a 1D function as

$g(x) = \int_{-\infty}^{\infty} g(\xi)\delta(x-\xi)d\xi$. To gain intuition, consider the approximation

$$g(x) \approx \sum_{n=-\infty}^{\infty} g(n\Delta x)\frac{1}{\Delta x}\Pi\left(\frac{x-n\Delta x}{\Delta x}\right)\Delta x.$$



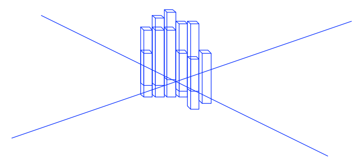
Representation of 2D Function

Similarly, we can write a 2D function as

$g(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\xi,\eta)\delta(x-\xi, y-\eta)d\xi d\eta$.

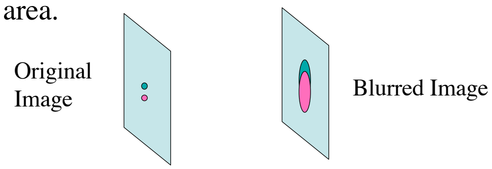
To gain intuition, consider the approximation

$$g(x,y) \approx \sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty} g(n\Delta x, m\Delta y)\frac{1}{\Delta x}\Pi\left(\frac{x-n\Delta x}{\Delta x}\right)\frac{1}{\Delta y}\Pi\left(\frac{y-m\Delta y}{\Delta y}\right)\Delta x\Delta y.$$

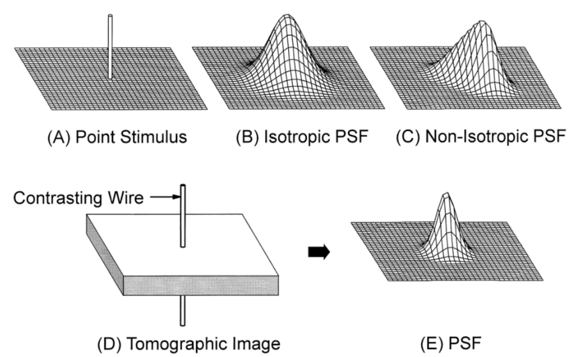


Impulse Response

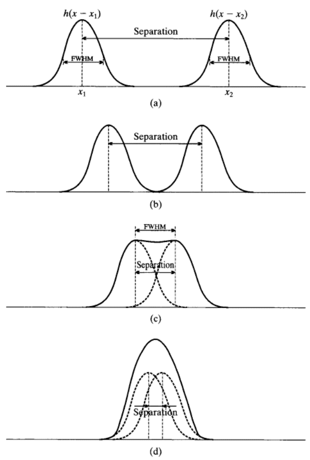
Intuition: the impulse response is the response of a system to an input of infinitesimal width and unit area.



Since any input can be thought of as the weighted sum of impulses, a linear system is characterized by its impulse response(s).



Bushberg et al 2001



Full Width Half Maximum (FWHM) is a measure of resolution.

Figure 3.6
An example of the effect of system resolution on the ability to differentiate two points. The FWHM equals the minimum distance that the two points must be separated in order to be distinguishable.

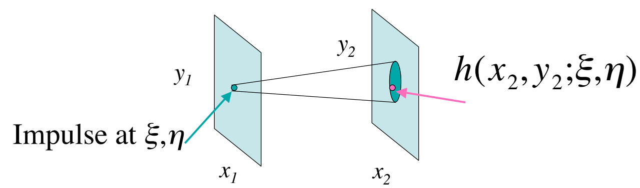
Prince and Link 2005

Impulse Response

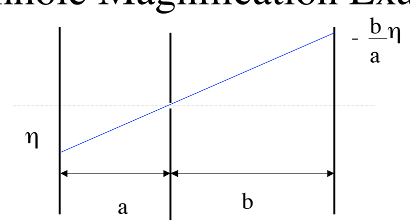
The impulse response characterizes the response of a system over all space to a Dirac delta impulse function at a certain location.

$h(x_2;\xi) = L[\delta(x_1 - \xi)]$ 1D Impulse Response

$h(x_2,y_2;\xi,\eta) = L[\delta(x_1 - \xi, y_1 - \eta)]$ 2D Impulse Response



Pinhole Magnification Example



In this example, an impulse at (ξ,η) will yield an impulse at $(m\xi, m\eta)$ where $m = -b/a$.

Thus, $h(x_2,y_2;\xi,\eta) = L[\delta(x_1 - \xi, y_1 - \eta)] = \delta(x_2 - m\xi, y_2 - m\eta)$.