Computing Transforms

\[ F(\delta(x)) = \int_{-\infty}^{\infty} \delta(x)e^{-j2\pi k x} \, dx = 1 \]

\[ F(\delta(x-x_0)) = \int_{-\infty}^{\infty} \delta(x-x_0)e^{-j2\pi k x} \, dx = e^{-j2\pi k x_0} \]

\[ F(1) = \int_{-\infty}^{\infty} e^{-j2\pi k x} \, dx = \frac{\sin(\pi x)}{\pi x} = \text{sinc}(x) \]

Similarly,

\[ F\{e^{j2\pi k x}\} = \delta(k - k_0) \]

\[ F\{\cos2\pi k x\} = \frac{1}{2} (\delta(k - k_0) + \delta(k + k_0)) \]

\[ F\{\sin2\pi k x\} = \frac{1}{2j} (\delta(k - k_0) - \delta(k + k_0)) \]
Examples

\[ g(x, y) = 1 + e^{j2\pi xy} \]
\[ G(k_x, k_y) = \delta(k_x) + \delta(k_x + a) \delta(k_y) \]

\[ g(x, y) = 1 + e^{j2\pi y} \]
\[ G(k_x, k_y) = \delta(k_x) + \delta(k_x + a) \delta(k_y - a) \]

\[ g(x, y) = \cos(2\pi(ax + by)) \]
\[ G(k_x, k_y) = \frac{1}{2} \delta(k_x - a) \delta(k_y - b) + \frac{1}{2} \delta(k_x + a) \delta(k_y + b) \]

Basic Properties

**Linearity**
\[ F[ag(x, y) + bh(x, y)] = aG(k_x, k_y) + bH(k_x, k_y) \]

**Scaling**
\[ F[g(ax, by)] = \frac{1}{ab} \begin{pmatrix} k_x \\ k_y \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} \]

**Shift**
\[ F[g(x - a, y - b)] = G(k_x, k_y) e^{-j2\pi(k_x a + k_y b)} \]

**Modulation**
\[ F[g(x, y) e^{j2\pi(ax + by)}] = G(k_x - a, k_y - b) \]
**Linearity**

The Fourier Transform is linear.

\[
F\{ag(x) + bh(x)\} = aG(k_x) + bH(k_y)
\]

\[
F\{ag(x, y) + bh(x, y)\} = aG(k_x, k_y) + bH(k_x, k_y)
\]

**Scaling Theorem**

\[
F\{g(ax)\} = \frac{1}{|a|}G\left(\frac{k_x}{a}\right)
\]

\[
F\{g(ax, by)\} = \frac{1}{|ab|}G\left(\frac{k_x}{a}, \frac{k_y}{b}\right)
\]

**Separable Functions**

\(g(x, y)\) is said to be a separable function if it can be
written as \(g(x, y) = g_x(x)g_y(y)\)

The Fourier Transform is then separable as well.

\[
G(k_x, k_y) = \iint g(x, y)e^{-j2\pi(k_xx+k_xy)}
\]

\[
= \int g_x(x)e^{-j2\pi k_xx}dx \int g_y(y)e^{-j2\pi k_yy}dy
\]

\[
= G_x(k_x)G_y(k_y)
\]

**Example (sinc/rect)**

Example
\(g(x, y) = \Pi(x)\Pi(y)\)
\(G(k_x, k_y) = \text{sinc}(k_x)\text{sinc}(k_y)\)

\[
= \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{1}{2}
\]
Example (sinc/rect)

\[ g(x, y) = \delta(x, y) = \delta(x)\delta(y) \]
\[ G(k_x, k_y) = 1 \]

Examples

\[ g(x, y) = \delta(x) \]
\[ G(k_x, k_y) = \delta(k_y) \]

Duality

Note the similarity between these two transforms

\[ F\{e^{j2\pi ak}\} = \delta(k_y - a) \]
\[ F\{\delta(x - a)\} = e^{-jk_0x} \]

These are specific cases of duality

\[ F\{G(x)\} = g(-k_y) \]

Application of Duality

\[ F\{\text{sinc}(x)\} = \int_{-\infty}^{\infty} \frac{\sin \pi k_x}{\pi k_x} e^{-j2\pi k_x a} \ dx = ?? \]

Recall that \( F\{\Pi(x)\} = \text{sinc}(k_x) \).
Therefore from duality, \( F\{\text{sinc}(x)\} = \Pi(-k_x) = \Pi(k_x) \)
**Shift Theorem**

\[ F\{g(x-a)\} = G(k_x)e^{-j2\pi ak_x} \]

\[ F\{g(x-a,y-b)\} = G(k_x,k_y)e^{-j2\pi(k_xa+k_yb)} \]

Shifting the function doesn’t change its spectral content, so the magnitude of the transform is unchanged. Each frequency component is shifted by \(a\). This corresponds to a relative phase shift of

\[-2\pi a/(\text{spatial period}) = -2\pi ak_x\]

For example, consider \(e^{j2\pi k_x x}\). Shifting this by \(a\) yields

\(e^{j2\pi k_x (x-a)} = e^{j2\pi k_x x}e^{-j2\pi ak_x}\)

**Modulation Example**

Amplitude Modulation (e.g. AM Radio)

\[ g(t) \rightarrow 2g(t) \cos(2\pi f_0 t) \]

\[ 2 \cos(2\pi f_0 t) \]

\[ G(f) \]

\[ G(f-f_0) + G(f+f_0) \]

**Modulation**

\[ F\{g(x)e^{j2\pi k_0 x}\} = G(k_x) * \delta(k_x - k_0) = G(k_x - k_0) \]

\[ F\{g(x)\cos(2\pi k_0 x)\} = \frac{1}{2} G(k_x - k_0) + \frac{1}{2} G(k_x + k_0) \]

\[ F\{g(x)\sin(2\pi k_0 x)\} = \frac{1}{2j} G(k_x - k_0) - \frac{1}{2j} G(k_x + k_0) \]
The fundamental nature of the convolution theorem may be better understood by observing that the complex exponentials are eigenfunctions of the convolution operator.

\[ e^{j2\pi k x} \]

The response of a linear shift invariant system to a complex exponential is simply the exponential multiplied by the FT of the system’s impulse response.

\[ z(x) = g(x) \ast e^{j2\pi k x} = \int_{-\infty}^{\infty} g(u) e^{j2\pi k (x-u)} du \]

\[ = G(k_x) e^{j2\pi k x} \]
MTF = Fourier Transform of PSF

Convolution/Multiplication

Now consider an arbitrary input \( h(x) \).

Recall that we can express \( h(x) \) as the integral of weighted complex exponentials.

Each of these exponentials is weighted by \( G(k_x) \) so that the response may be written as

\[
 z(x) = \int_{-\infty}^{\infty} G(k_x) H(k_x) e^{j 2\pi k_x x} dk_x
\]

Convolution/Modulation

Theorem

\[
 F \{ g(x) \ast h(x) \} = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} g(u) \ast h(x-u) du \right] e^{-j 2\pi k_x x} dx
\]

= \int_{-\infty}^{\infty} g(u) \int_{-\infty}^{\infty} h(x-u) e^{-j 2\pi k_x x} dx du

= \int_{-\infty}^{\infty} g(u) H(k_x) e^{-j 2\pi k_x u} du

= G(k_x) H(k_x)

Convolution in the spatial domain transforms into multiplication in the frequency domain. Dual is modulation

\[
 F \{ g(x) h(x) \} = G(k_x) \ast H(k_x)
\]

2D Convolution/Multiplication

Convolution

\[
 F \{ g(x,y) \ast h(x,y) \} = G(k_x,k_y) H(k_x,k_y)
\]

Multiplication

\[
 F \{ g(x,y) h(x,y) \} = G(k_x,k_y) \ast H(k_x,k_y)
\]
Application of Convolution Thm.

\[ \Lambda(x) = \begin{cases} |1 - |x| & |x| < 1 \\ 0 & \text{otherwise} \end{cases} \]

\[ F(\Lambda(x)) = \int_{-1}^{1} (1 - |x|) e^{-j\pi k x} dx = ?? \]

Application of Convolution Thm.

\[ \Lambda(x) = \Pi(x) * \Pi(x) \]

\[ F(\Lambda(x)) = \text{sinc}^2(k_x) \]

Convolution Example

Response of an Imaging System

\[ g(x,y) \quad h_1(x,y) \quad h_2(x,y) \quad h_3(x,y) \]

\[ Z(k_x, k_y) \]

\[ z(x,y) = g(x,y) * h_1(x,y) * h_2(x,y) * h_3(x,y) \]

\[ Z(k_x, k_y) = G(k_x, k_y) H_1(k_x, k_y) H_2(k_x, k_y) H_3(k_x, k_y) \]
System MTF = Product of MTFs of Components

Useful Approximation

$$FWHM_{\text{system}} = \sqrt{FWHM_1^2 + FWHM_2^2 + \cdots + FWHM_N^2}$$

Example

$$FWHM_1 = 1\text{mm}$$
$$FWHM_2 = 2\text{mm}$$
$$FWHM_{\text{system}} = \sqrt{5} = 2.24\text{mm}$$