Torque

For a non-spinning magnetic moment, the torque will try to align the moment with magnetic field (e.g. compass needle)

Precession

Analogous to motion of a gyroscope
Precesses at an angular frequency of
\[ \omega = \gamma B \]
This is known as the Larmor frequency.
Magnetization Vector

Vector sum of the magnetic moments over a volume.

For a sample at equilibrium in a magnetic field, the transverse components of the moments cancel out, so that there is only a longitudinal component.

Equation of motion is the same form as for individual moments.

$$M = \frac{1}{V} \sum_{\text{protons}} \mu_i$$

$$\frac{dM}{dt} = \gamma M \times B$$

Free precession about static field

$$\frac{dM}{dt} = M \times \gamma B$$

$$= \begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
M_x & M_y & M_z \\
B_x & B_y & B_z
\end{vmatrix}$$

$$= \gamma \begin{vmatrix}
\hat{i}(B_x M_y - B_y M_x) & \hat{j}(B_x M_z - B_z M_x) & \hat{k}(B_y M_z - B_z M_y)
\end{vmatrix}$$

RF Excitation

Free precession about static field

$$\frac{dM_x}{dt} = \gamma \left[B_x M_y - B_y M_x\right]$$

$$\frac{dM_y}{dt} = \gamma \left[B_x M_z - B_z M_x\right]$$

$$\frac{dM_z}{dt} = \gamma \left[-B_x - B_y B_z\right]$$

$$\begin{bmatrix}
0 & B_x & -M_y \\
-B_x & 0 & B_z \\
B_y & -B_z & 0
\end{bmatrix}$$

Hansen 2009

http://www.drcnr.dk/main/content/view/213/74/
Precession

\[
\begin{bmatrix}
\frac{dM_x}{dt} \\
\frac{dM_y}{dt} \\
\frac{dM_z}{dt}
\end{bmatrix} = 
\gamma
\begin{bmatrix}
B_0 & 0 & 0 \\
0 & -B_0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
\]

Useful to define \( M = M_x + jM_y \)

\[
dM/dt = d/dt(M_x + iM_y) = -j\gamma B_0 M
\]

Solution is a time-varying phasor

\[ M(t) = M(0)e^{-j\gamma B_0 t} = M(0)e^{-j\omega_0 t} \]

Question: which way does this rotate with time?

Gyromagnetic Ratios

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>Spin</th>
<th>Magnetic Moment</th>
<th>( \gamma/(2\pi) ) (MHz/Tesla)</th>
<th>Abundance</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^1\text{H})</td>
<td>1/2</td>
<td>2.793</td>
<td>42.58</td>
<td>88 M</td>
</tr>
<tr>
<td>(^{23}\text{Na})</td>
<td>3/2</td>
<td>2.216</td>
<td>11.27</td>
<td>80 mM</td>
</tr>
<tr>
<td>(^{31}\text{P})</td>
<td>1/2</td>
<td>1.131</td>
<td>17.25</td>
<td>75 mM</td>
</tr>
</tbody>
</table>

Source: Haacke et al., p. 27

Larmor Frequency

\[ \omega = \gamma B \quad \text{Angular frequency in rad/sec} \]

\[ f = \gamma B / (2 \pi) \quad \text{Frequency in cycles/sec or Hertz, Abbreviated Hz} \]

For a 1.5 T system, the Larmor frequency is 63.86 MHz which is 63.86 million cycles per second. For comparison, KPBS-FM transmits at 89.5 MHz.

Note that the earth’s magnetic field is about 50 µT, so that a 1.5T system is about 30,000 times stronger.

Notation and Units

1 Tesla = 10,000 Gauss
Earth's field is about 0.5 Gauss
0.5 Gauss = 0.5x10\(^{-4}\) T = 50 µT

\[ \gamma = 26752 \text{ radians/second/Gauss} \]
\[ \gamma = \gamma / 2 \pi = 4258 \text{ Hz/Gauss} \]
\[ = 42.58 \text{ MHz/Tesla} \]
Gradients

Spins precess at the Larmor frequency, which is proportional to the local magnetic field. In a constant magnetic field $B_z = B_0$, all the spins precess at the same frequency (ignoring chemical shift).

Gradient coils are used to add a spatial variation to $B_z$ such that $B_z(x, y, z) = B_0 + \Delta B_z(x, y, z)$. Thus, spins at different physical locations will precess at different frequencies.

Gradient Fields

$$B_z(x, y, z) = B_0 + \frac{\partial B_z}{\partial x} x + \frac{\partial B_z}{\partial y} y + \frac{\partial B_z}{\partial z} z$$

$$= B_0 + G_x x + G_y y + G_z z$$

$$G_z = \frac{\partial B_z}{\partial z} > 0 \quad G_y = \frac{\partial B_z}{\partial y} > 0$$

Interpretation

$$\Delta B_z(x) = G_x x$$

Spins precess at $\gamma B_0 + \gamma G_x x$ (slower)

$$M(t) = M(0) e^{-j(\omega_0 + \Delta \omega)t}$$

Spins precess at $\gamma B_0$ (faster)

$$M(t) = M(0) e^{-j(\omega_0 + \gamma G_x x)t}$$

Spins precess at $\gamma B_0 + \gamma G_x x$

$$M(t) = M(0) e^{-j(\omega_0 + \gamma G_x x)t}$$
Rotating Frame of Reference
Reference everything to the magnetic field at isocenter.

There is nothing that nuclear spins will not do for you, as long as you treat them as human beings.
Erwin Hahn

Phasor Diagram

\[ G(k_x) = \int_{-\infty}^{\infty} g(x) \exp(-j2\pi k_x x) dx \]
\[ \theta = -2\pi k_x x \]
\[ k_x = 1; x = 0 \]
\[ 2\pi k_x x = 0 \]
\[ 2\pi k_x x = \pi / 2 \]
\[ x = 1 / 4 \]
\[ 2\pi k_x x = \pi \]
\[ x = 1 / 2 \]
\[ 2\pi k_x x = 3\pi / 4 \]
\[ x = 3 / 2 \]

\[ \theta = -\pi / 2 \]
\[ \theta = 0 \]
\[ \theta = \pi \]
\[ \theta = 3\pi / 2 \]

Interpretation

\[ \Delta B_z(x) = g(x) \]

Slower \[ \Delta x \]
Faster \[ 2\Delta x \]

\[ \exp(-j2\pi \frac{0}{8\Delta x}) x \]
\[ \exp(-j2\pi \frac{1}{8\Delta x}) x \]
\[ \exp(-j2\pi \frac{2}{8\Delta x}) x \]
\( k_x = 0; \ k_y = 0 \)

\( k_x = 0; \ k_y \neq 0 \)

Phase with time-varying gradient
K-space trajectory

\[ G_x(t) \]

\[ G_y(t) \]

\[ k_x \]

\[ k_y \]
Spin-Warp

\[ G_x(t) \]

\[ G_y(t) \]

k-space

Image space

k-space

Fourier Transform

k-space
Spin-Warp Pulse Sequence

Spin-Warp

Hanson 2009
Gradient Fields

Define
\[ \vec{G} = G_x \hat{i} + G_y \hat{j} + G_z \hat{k} \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \]

So that
\[ G_x x + G_y y + G_z z = \vec{G} \cdot \vec{r} \]

Also, let the gradient fields be a function of time. Then the z-directed magnetic field at each point in the volume is given by:
\[ B_z(\vec{r}, t) = B_0 + \vec{G}(t) \cdot \vec{r} \]

Static Gradient Fields

In a uniform magnetic field, the transverse magnetization is given by:
\[ M(t) = M(0)e^{-j\omega_0 t}e^{-t/T_2} \]

In the presence of non time-varying gradients we have
\[ M(\vec{r}) = M(\vec{r}, 0)e^{-j\gamma B_z(\vec{r}) t}e^{-t/T_2(\vec{r})} = M(\vec{r}, 0)e^{-j\gamma B_0 t}e^{-t/T_2(\vec{r})} = M(\vec{r}, 0)e^{-j\omega_0 t}e^{-t/T_2(\vec{r})} \]

Time-Varying Gradient Fields

In the presence of time-varying gradients the frequency as a function of space and time is:
\[ \omega(\vec{r}, t) = \gamma B_z(\vec{r}, t) \]
\[ = \gamma B_0 + \gamma \vec{G}(t) \cdot \vec{r} \]
\[ = \omega_0 + \Delta \omega(\vec{r}, t) \]

Phasors

\[ e^{\theta} = \cos \theta + j \sin \theta \]

\[ \theta = 0 \quad \theta = -\pi/2 \quad \theta = \pi \quad \theta = \pi/2 \]
Phase

Phase = angle of the magnetization phasor
Frequency = rate of change of angle (e.g. radians/sec)
Phase = time integral of frequency
\[ \phi_{\rho r, t}(t) = -\int_0^t \omega(\rho r, \tau) d\tau = -\omega_0 t + \Delta \phi(\rho r, t) \]

Where the incremental phase due to the gradients is
\[ \Delta \phi(\rho r, t) = -\int_0^t \Delta \omega(\rho r, \tau) d\tau \]

Time-Varying Gradient Fields

The transverse magnetization is then given by
\[ M(\rho r, t) = M(\rho, 0) e^{-jT_2/2} e^{j\phi(\rho r, t)} \]
\[ = M(\rho, 0) e^{-jT_2/2} e^{j\omega_0 t} \exp \left(-j\int_0^t \Delta \omega(\rho r, \tau) d\tau \right) \]
\[ = M(\rho, 0) e^{-jT_2/2} e^{j\omega_0 t} \exp \left(-j\gamma \int_\tau^t G(\rho r) \cdot \hat{r} d\tau \right) \]

Phase with constant gradient

\[ \Delta \phi(\rho r, t) = -\int_0^t \Delta \omega(\rho r, \tau) d\tau \]
\[ \Delta \phi(\rho r, t) = -\int_0^t \Delta \omega(\rho r, \tau) d\tau \]
\[ \Delta \phi(\rho r, t) = -\Delta \omega(\rho r, t) \]
if \( \Delta \omega \) is non-time varying.

Signal Equation

Signal from a volume
\[ s(t) = \int V M(\rho r, t) dV \]
\[ = \int \int \int V(x, y, z, 0) e^{-jT_2/2} e^{j\omega_0 t} \exp \left(-j\gamma \int_\tau^t G(\rho r) \cdot \hat{r} d\tau \right) dx dy dz \]

For now, consider signal from a slice along \( z \) and drop the \( T_2 \) term. Define
\[ m(x, y) = \int_{-\Delta z/2}^{\Delta z/2} M(\rho r, t) dz \]

To obtain
\[ s(t) = \int \int m(x, y) e^{-j\omega_0 t} \exp \left(-j\gamma \int_\tau^t G(\rho r) \cdot \hat{r} d\tau \right) dx dy \]
**Signal Equation**

Demodulate the signal to obtain

\[ s(t) = e^{j\omega_0 t} \]
\[ s(t) = \int \int m(x, y) \exp \left\{ -j \int G_x \cdot \mathbf{r} \, dt \right\} dx \, dy \]
\[ s(t) = \int \int m(x, y) \exp \left\{ -j \int \left[ G_x(t)x + G_y(t)y \right] \, dt \right\} dx \, dy \]

Where

\[ k_x(t) = \frac{\gamma}{2\pi} \int G_x(t) \, dt \]
\[ k_y(t) = \frac{\gamma}{2\pi} \int G_y(t) \, dt \]

**MR signal is Fourier Transform**

\[ s(t) = \int \int m(x, y) \exp \left\{ -j2\pi \left( k_x(t)x + k_y(t)y \right) \right\} dx \, dy \]
\[ s(t) = M(k_x(t), k_y(t)) \]
\[ s(t) = F[m(x, y)]_{(k_x(t), k_y(t))} \]

**Recap**

- Frequency = rate of change of phase.
- Higher magnetic field -> higher Larmor frequency -> phase changes more rapidly with time.
- With a constant gradient \( G_x \), spins at different x locations precess at different frequencies -> spins at greater x-values change phase more rapidly.
- With a constant gradient, distribution of phases across x locations changes with time. (phase modulation)
- More rapid change of phase with x -> higher spatial frequency \( k_x \)

**K-space**

At each point in time, the received signal is the Fourier transform of the object

\[ s(t) = M(k_x(t), k_y(t)) = F[m(x, y)]_{(k_x(t), k_y(t))} \]

evaluated at the spatial frequencies:

\[ k_x(t) = \frac{\gamma}{2\pi} \int G_x(t) \, dt \]
\[ k_y(t) = \frac{\gamma}{2\pi} \int G_y(t) \, dt \]

Thus, the gradients control our position in k-space. The design of an MRI pulse sequence requires us to efficiently cover enough of k-space to form our image.
K-space trajectory

Units

Spatial frequencies \(k_x, k_y\) have units of 1/distance. Most commonly, 1/cm

Gradient strengths have units of (magnetic field)/distance. Most commonly \(G/cm\) or \(mT/m\)

\(\gamma/(2\pi)\) has units of \(Hz/G\) or \(Hz/Tesla\).

\[
k_x(t) = \frac{\gamma}{2\pi} \int_0^t G_x(\tau)d\tau
\]

\[
= \left[ Hz/Gauss \right] \left[ Gauss/cm \right] \left[ sec \right] \\
= \left[ Hz/1/cm \right] \\
= \left[ 1/cm \right]
\]

Example

\[
k_x(t_2) = \frac{\gamma}{2\pi} \int_0^{t_2} G_x(\tau)d\tau \\
= 4257 \text{ Hz/G \cdot 1G/cm \cdot 0.235 \times 10^{-3}} \text{ s} \\
= 1 \text{ cm}^{-1}
\]

TT. Lin, BE280A, UCSD Fall 2012