Spin

- Intrinsic angular momentum of elementary particles -- electrons, protons, neutrons.
- Spin is quantized. Key concept in Quantum Mechanics.

Magnetic Moment and Angular Momentum

A charged sphere spinning about its axis has angular momentum and a magnetic moment. This is a classical analogy that is useful for understanding quantum spin, but remember that it is only an analogy!

Relation: \( \mu = \gamma S \) where \( \gamma \) is the gyromagnetic ratio and \( S \) is the spin angular momentum.

Nuclear Spin Rules

<table>
<thead>
<tr>
<th>Number of Protons</th>
<th>Number of Neutrons</th>
<th>Spin</th>
<th>Examples</th>
</tr>
</thead>
<tbody>
<tr>
<td>Even</td>
<td>Even</td>
<td>0</td>
<td>12C, 16O</td>
</tr>
<tr>
<td>Even</td>
<td>Odd</td>
<td>( j/2 )</td>
<td>17O</td>
</tr>
<tr>
<td>Odd</td>
<td>Even</td>
<td>( j/2 )</td>
<td>1H, 23Na, 31P</td>
</tr>
<tr>
<td>Odd</td>
<td>Odd</td>
<td>( j )</td>
<td>2H</td>
</tr>
</tbody>
</table>
Classical Magnetic Moment

\[ \vec{\mu} = IA\hat{n} \]

Energy in a Magnetic Field

\[ E = -\vec{\mu} \cdot \vec{B} = -\mu_z B \]

Lorentz Force

Maximum Energy State

Minimum Energy State

Magnetic Field Units

1 Tesla = 10,000 Gauss

Earth’s field is about 0.5 Gauss

0.5 Gauss = 0.5 \times 10^{-4} \, T = 50 \, \mu T

Earth's Magnetic Field

www.qi-whiz.com/images/Earth-magnetic-field.jpg
Boltzmann Distribution

\[ \frac{N_j}{N} = \mathcal{P}(\epsilon_j) = \frac{e^{-\epsilon_j/(kT)}}{Q} \]

Boltzmann Distribution

Equilibrium Magnetization

\[ M_0 = N \langle \mu \rangle \]
\[ = N\mu B / (kT) \]
\[ = N\gamma^2 h^2 B / (4kT) \]

M = number of nuclear spins per unit volume
Magnetization is proportional to applied field.

Hansen 2009
**MRI System**

Nova 32 channel

Siemens 32 channel

**MRI Gradients**

**Torque**

For a non-spinning magnetic moment, the torque will try to align the moment with magnetic field (e.g. compass needle)

\[
N = \mu \times B
\]
Precession

Torque

\[ \mathbf{N} = \mu \times \mathbf{B} \]

Change in Angular momentum

\[ \frac{d\mathbf{S}}{dt} = \mathbf{N} \]

Relation between magnetic moment and angular momentum

\[ \frac{d\mu}{dt} = \gamma \mathbf{S} \times \mathbf{B} \]

Analogous to motion of a gyroscope

Precesses at an angular frequency of

\[ \omega = \gamma B \]

This is known as the Larmor frequency.

Magnetization Vector

Vector sum of the magnetic moments over a volume.

For a sample at equilibrium in a magnetic field, the transverse components of the moments cancel out, so that there is only a longitudinal component.

Equation of motion is the same form as for individual moments.

\[ \mathbf{M} = \frac{1}{V} \sum_{\text{protons}} \mu_i \]

\[ \frac{d\mathbf{M}}{dt} = \gamma \mathbf{M} \times \mathbf{B} \]

Hansen 2009

RF Excitation

http://www.drcmr.dk/main/content/view/213/74/
Free precession about static field

\[ \frac{dM}{dt} = M \times \gamma B \]

- Here, \( M \) represents the magnetic moment,
- \( \gamma \) is the gyromagnetic ratio,
- \( B \) is the magnetic field.

\[ dM_x = \gamma M_x B_z dt \]
\[ dM_y = -\gamma B_z M_y dt \]
\[ dM_z = -\gamma B_y M_z dt \]

Precession

\[
\begin{bmatrix}
\frac{dM_x}{dt} \\
\frac{dM_y}{dt} \\
\frac{dM_z}{dt}
\end{bmatrix} = \gamma
\begin{bmatrix}
0 & B_0 & 0 \\
-B_0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
M_x \\
M_y \\
M_z
\end{bmatrix}
\]

Useful to define

\[ M = M_x + jM_y \]

\[ \frac{dM}{dt} = \frac{d}{dt}(M_x + iM_y) \]

\[ = -j\gamma B_0 M \]

Solution is a time-varying phasor

\[ M(t) = M(0)e^{-j\omega_0 t} = M(0)e^{-j\omega_0 t} \]

Gyromagnetic Ratios

<table>
<thead>
<tr>
<th>Nucleus</th>
<th>Spin</th>
<th>Magnetic Moment</th>
<th>( \gamma(2\pi) ) (MHz/Tesla)</th>
<th>Abundance</th>
</tr>
</thead>
<tbody>
<tr>
<td>(^1\text{H})</td>
<td>1/2</td>
<td>2.793</td>
<td>42.58</td>
<td>88 M</td>
</tr>
<tr>
<td>(^{23}\text{Na})</td>
<td>3/2</td>
<td>2.216</td>
<td>11.27</td>
<td>80 mM</td>
</tr>
<tr>
<td>(^{31}\text{P})</td>
<td>1/2</td>
<td>1.131</td>
<td>17.25</td>
<td>75 mM</td>
</tr>
</tbody>
</table>

Source: Haacke et al., p. 27

Question: which way does this rotate with time?
**Larmor Frequency**

\[ \omega = \gamma B \quad \text{Angular frequency in rad/sec} \]

\[ f = \frac{\gamma B}{2\pi} \quad \text{Frequency in cycles/sec or Hertz, Abbreviated Hz} \]

For a 1.5 T system, the Larmor frequency is 63.86 MHz which is 63.86 million cycles per second. For comparison, KPBS-FM transmits at 89.5 MHz.

Note that the earth’s magnetic field is about 50 \( \mu T \), so that a 1.5T system is about 30,000 times stronger.

**Notation and Units**

1 Tesla = 10,000 Gauss

Earth's field is about 0.5 Gauss

0.5 Gauss = 0.5x10\(^{-4}\) T = 50 \( \mu T \)

\[ \gamma = 26752 \text{ radians/second/Gauss} \]

\[ \gamma = \frac{\gamma}{2\pi} = 4258 \text{ Hz/Gauss} \]

= 42.58 MHz/Tesla

**Gradients**

Spins precess at the Larmor frequency, which is proportional to the local magnetic field. In a constant magnetic field \( B_z = B_0 \), all the spins precess at the same frequency (ignoring chemical shift).

Gradient coils are used to add a spatial variation to \( B_z \) such that \( B_z(x,y,z) = B_0 + \Delta B_z(x,y,z) \). Thus, spins at different physical locations will precess at different frequencies.

**MRI System**

Simplified Drawing of Basic Instrumentation.

Body lies on table encompassed by coils for static field \( B_0 \), gradient fields (two of three shown), and radiofrequency field \( B_1 \).

Image, caption: copyright Nishimura, Fig. 3.15
**Interpretation**

\[ \Delta B_z(x) = G_x x \]

- Spins Precess at \( \gamma B_0 - \gamma G_x x \) (slower)
- Spins Precess at \( \gamma B_0 + \gamma G_x x \) (faster)

**Gradient Fields**

\[ B_z(x, y, z) = B_0 + \frac{\partial B_z}{\partial x} x + \frac{\partial B_z}{\partial y} y + \frac{\partial B_z}{\partial z} z \]

\[ = B_0 + G_x x + G_y y + G_z z \]

\[ G_z = \frac{\partial B_z}{\partial z} > 0 \]

\[ G_y = \frac{\partial B_z}{\partial y} > 0 \]

**Rotating Frame of Reference**

Reference everything to the magnetic field at isocenter.

**Spins**

*There is nothing that nuclear spins will not do for you, as long as you treat them as human beings.*

Erwin Hahn
Phasors

\[ \theta = 0 \]
\[ \theta = -\pi/2 \]
\[ \theta = \pi \]
\[ \theta = \pi/2 \]

Interpretation

\[ \Delta B(x) = G(x) \]

Faster

Slower
Fig 3.12 from Nishimura

\[ k_x = 0; \quad k_y = 0 \]

\[ k_x = 0; \quad k_y \neq 0 \]

Hanson 2009

k-space

Image space

k-space

Fourier Transform
2D Fourier Transform

Fourier Transform

\[ G(k_x, k_y) = \mathcal{F}\{g(x, y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{-j2\pi(k_x x + k_y y)} \, dx \, dy \]

Inverse Fourier Transform

\[ g(x, y) = \mathcal{F}^{-1}\{G(k_x, k_y)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(k_x, k_y) e^{j2\pi(k_x x + k_y y)} \, dk_x \, dk_y \]