2D Fourier Transform

Fourier Transform

\[ G(k_x, k_y) = \mathcal{F}[g(x, y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y) e^{-j2\pi(k_x x + k_y y)} \, dx \, dy \]

Inverse Fourier Transform

\[ g(x, y) = \mathcal{F}^{-1}[G(k_x, k_y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} G(k_x, k_y) e^{j2\pi(k_x x + k_y y)} \, dk_x \, dk_y \]

Hanson 2009
Gradient Fields

Define
\[ \vec{G} = G_x \hat{i} + G_y \hat{j} + G_z \hat{k} \quad \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \]

So that
\[ G_x x + G_y y + G_z z = \vec{G} \cdot \vec{r} \]

Also, let the gradient fields be a function of time. Then the $z$-directed magnetic field at each point in the volume is given by:
\[ B_z(\vec{r}, t) = B_0 + \vec{G}(t) \cdot \vec{r} \]

Static Gradient Fields

In a uniform magnetic field, the transverse magnetization is given by:
\[ M(t) = M(0)e^{-j\omega_0 t}e^{-t/T_2} \]

In the presence of non time-varying gradients we have
\[
\begin{align*}
M(\vec{r}) &= M(\vec{r}, 0)e^{-j[\vec{B}_0(\vec{r}) \cdot \vec{r}]}e^{-t/T_2(\vec{r})} \\
&= M(\vec{r}, 0)e^{-j[\vec{B}_0(\vec{r}) \cdot \vec{r}]}e^{-t/T_2(\vec{r})} \\
&= M(\vec{r}, 0)e^{-J_{0z} t}e^{-j\vec{G}(t) \cdot \vec{r} e^{-t/T_2(\vec{r})}}
\end{align*}
\]
**Phasors**

Phase = angle of the magnetization phasor  
Frequency = rate of change of angle (e.g. radians/sec)  
Phase = time integral of frequency  

\[
\Delta \phi(\vec{r},t) = -\int_0^t \Delta \omega(\vec{r},\tau) d\tau 
\]

Where the incremental phase due to the gradients is

\[
\Delta \phi(\vec{r},t) = -\int_0^t \Delta \omega(\vec{r},\tau) d\tau = -\gamma \vec{G}(\vec{r},\tau) \cdot \hat{r} d\tau
\]

**Phase with constant gradient**

In the presence of time-varying gradients the frequency as a function of space and time is:

\[
\omega(\vec{r},t) = \gamma B_z(\vec{r},t) = \omega_0 + \Delta \omega(\vec{r},t)
\]

**Time-Varying Gradient Fields**

Phase

Frequency = rate of change of angle (e.g. radians/sec)
Phase = time integral of frequency

\[
q(\vec{r},t) = -\int_0^t \omega(\vec{r},\tau) d\tau = -\omega_0 t + \Delta q(\vec{r},t)
\]

Where the incremental phase due to the gradients is

\[
\Delta q(\vec{r},t) = -\int_0^t \Delta \omega(\vec{r},\tau) d\tau = -\int_0^t \gamma \vec{G}(\vec{r},\tau) \cdot \hat{r} d\tau
\]
Phase with time-varying gradient

![Diagram of phase with time-varying gradient]

**Signal Equation**

Signal from a volume

\[ s_j(t) = \int_M M(\vec{r}, t) dV \]

\[ = \int_M \int_{-a}^{a/2} m(x, y) e^{-j\gamma f \int_{-T2}^{t} G(\tau) \cdot \vec{r} d\tau} \, dx \, dy \]

For now, consider signal from a slice along \( z \) and drop the \( T_2 \) term. Define \( m(x, y) = \int_{-a}^{a/2} M(\vec{r}, t) \, dz \)

To obtain \( s_j(t) = \int_M m(x, y) e^{-j\omega_o t} \exp\left(-j\gamma \int_{-T2}^{t} G(\tau) \cdot \vec{r} d\tau\right) \, dx \, dy \)

**Time-Varying Gradient Fields**

The transverse magnetization is then given by

\[ M(\vec{r}, t) = M(\vec{r}, 0) e^{-t/T_2} e^{i\omega_o t} \]

\[ = M(\vec{r}, 0) e^{-t/T_2} e^{-j\nu_o t} \exp\left(-j \int_0^t \Delta \omega(t, t') d\tau\right) \]

\[ = M(\vec{r}, 0) e^{-t/T_2} e^{-j\nu_o t} \exp\left(-j\gamma \int_0^t \hat{G}(\tau) \cdot \vec{r} d\tau\right) \]

**Signal Equation**

Demodulate the signal to obtain

\[ s(t) = e^{j\omega_o t} s_j(t) \]

\[ = \int_M \int_{-a}^{a/2} m(x, y) \exp\left(-j\gamma \int_{-T2}^{t} G(\tau) \cdot \vec{r} d\tau\right) \, dx \, dy \]

\[ = \int_M \int_{-a}^{a/2} m(x, y) \exp\left(-j\gamma \int_{-T2}^{t} \left[G_\gamma(\tau) x + G_\gamma(\tau) y\right] d\tau\right) \, dx \, dy \]

\[ = \int_M \int_{-a}^{a/2} m(x, y) \exp\left(-j2\pi \left(k_x(t) x + k_y(t) y\right)\right) \, dx \, dy \]

Where

\[ k_x(t) = \frac{\gamma}{2\pi} \int_0^t G_y(\tau) d\tau \]

\[ k_y(t) = \frac{\gamma}{2\pi} \int_0^t G_x(\tau) d\tau \]
MR signal is Fourier Transform

\[ s(t) = \int \int m(x,y) \exp\left(-j2\pi(k_x(t)x + k_y(t)y)\right) dx dy \]
\[ = M(k_x(t),k_y(t)) \]
\[ = F[m(x,y)]_{k_x(t),k_y(t)} \]

Recap

- Frequency = rate of change of phase.
- Higher magnetic field -> higher Larmor frequency -> phase changes more rapidly with time.
- With a constant gradient \( G_x \), spins at different x locations precess at different frequencies -> spins at greater x-values change phase more rapidly.
- With a constant gradient, distribution of phases across x locations changes with time. (phase modulation)
- More rapid change of phase with x -> higher spatial frequency \( k_x \)

K-space

At each point in time, the received signal is the Fourier transform of the object

\[ s(t) = M(k_x(t),k_y(t)) = F[m(x,y)]_{k_x(t),k_y(t)} \]

evaluated at the spatial frequencies:

\[ k_x(t) = \frac{\gamma}{2\pi} \int_0^t G_x(\tau) d\tau \]
\[ k_y(t) = \frac{\gamma}{2\pi} \int_0^t G_y(\tau) d\tau \]

Thus, the gradients control our position in k-space. The design of an MRI pulse sequence requires us to efficiently cover enough of k-space to form our image.
Units

Spatial frequencies \((k_x, k_y)\) have units of \(1/\text{distance}\). Most commonly, \(1/\text{cm}\)

Gradient strengths have units of \((\text{magnetic field})/\text{distance}\). Most commonly \(\text{G/cm} \text{ or mT/m}\)

\(\gamma/(2\pi)\) has units of \(\text{Hz/G} \text{ or Hz/Tesla}\).

\[
k_x(t) = \frac{\gamma}{2\pi} \int_0^t G_x(\tau) d\tau
\]

\[
= [\text{Hz/Gauss}] [\text{Gauss/cm}] [\text{sec}]
\]

\[
= [1/\text{cm}]
\]

Example

\[G_x(t) = 1 \text{ Gauss/cm}\]

\[
k_x(t_2) = \frac{\gamma}{2\pi} \int G_x(\tau) d\tau
\]

\[
= \frac{4257 \text{ Hz/G \cdot cm}}{0.235 \times 10^{-3} \text{ s}}
\]

\[
= 1 \text{ cm}^{-1}
\]

K-space trajectory

\[G_x(t)\]

\[t_1 \quad t_2 \quad t_3 \quad t_4\]

\[k_x(t_1) \quad k_x(t_2) \quad k_y(t_1) \quad k_y(t_2)\]
Spin-Warp

K-space trajectory

k-space

Fourier Transform
Spin-Warp Pulse Sequence

Spin-Warp
Sampling in k-space

Aliasing

Intuitive view of Aliasing
Fourier Sampling

Instead of sampling the signal, we sample its Fourier Transform

\[ F \left( \frac{k}{\Delta k_x} \right) \delta \left( n - \frac{k}{\Delta k_x} \right) \]

Nyquist Condition

To avoid overlap, \( 1/\Delta k_x > \text{FOV} \), or equivalently, \( \Delta k_x < 1/\text{FOV} \)
Aliasing occurs when $\frac{1}{\Delta k_x} < \text{FOV}$.

2D Comb Function

\[
\text{comb}(x, y) = \sum_{m=\infty}^{\infty} \sum_{n=\infty}^{\infty} \delta(x - m, y - n)
\]

\[
= \sum_{m=\infty}^{\infty} \sum_{n=\infty}^{\infty} \delta(x - m) \delta(y - n)
\]

\[
= \text{comb}(x) \text{comb}(y)
\]

Scaled 2D Comb Function

\[
\text{comb}(x/\Delta x, y/\Delta y) = \text{comb}(x/\Delta x) \text{comb}(y/\Delta y)
\]

\[
= \Delta x \Delta y \sum_{m=\infty}^{\infty} \sum_{n=\infty}^{\infty} \delta(x - m\Delta x) \delta(y - n\Delta y)
\]
2D k-space sampling

\[ G_s(k_x, k_y) = G(k_x, k_y) \frac{1}{\Delta k_x \Delta k_y} \text{comb} \left( \frac{k_x}{\Delta k_x}, \frac{k_y}{\Delta k_y} \right) \]

\[ = G(k_x, k_y) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(k_x - m\Delta k_x, k_y - n\Delta k_y) \]

\[ = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} G(m\Delta k_x, n\Delta k_y) \delta(k_x - m\Delta k_x, k_y - n\Delta k_y) \]

Nyquist Conditions

\[ \frac{1}{\Delta k_x} > \text{FOV}_x \]

\[ \frac{1}{\Delta k_y} > \text{FOV}_y \]