Spin-Warp Pulse Sequence

\[
\begin{align*}
G_x(t) & \quad G_y(t) \\
G_z(t) &
\end{align*}
\]
Sampling in k-space

Aliasing

Intuitive view of Aliasing

\[ \Delta B_z(x) = G_z x \]

\[ k_z = \frac{1}{\text{FOV}} \]

\[ k_z = \frac{2}{\text{FOV}} \]
Fourier Sampling

Instead of sampling the signal, we sample its Fourier Transform

\[
\text{Sample} \quad F^{-1}
\]

Fourier Sampling -- Inverse Transform

\[
\text{Sample} \quad \frac{1}{\Delta k_x} \text{comb}
\]

\[
\Delta k_x
\]

\[
\frac{1}{\Delta k_x} \text{comb}
\]

Nyquist Condition

To avoid overlap, \(1/\Delta k_x > \text{FOV}\), or equivalently, \(\Delta k_x < 1/\text{FOV}\)
Aliasing occurs when \( \frac{1}{\Delta k_x} < \text{FOV} \)

2D Comb Function

\[
\text{comb}(x,y) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m, y - n)
= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m) \delta(y - n)
= \text{comb}(x) \text{comb}(y)
\]

Scaled 2D Comb Function

\[
\text{comb}(x/\Delta x, y/\Delta y) = \text{comb}(x/\Delta x) \text{comb}(y/\Delta y)
= \Delta x \Delta y \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(x - m\Delta x) \delta(y - n\Delta y)
\]
2D k-space sampling

\[ G_s(k_x, k_y) = G(k_x, k_y) \frac{1}{\Delta k_x \Delta k_y} \text{comb}\left(\frac{k_x}{\Delta k_x}, \frac{k_y}{\Delta k_y}\right) \]

\[ = G(k_x, k_y) \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(k_x - m\Delta k_x, k_y - n\Delta k_y) \]

\[ = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} G(m\Delta k_x, n\Delta k_y) \delta(k_x - m\Delta k_x, k_y - n\Delta k_y) \]

Nyquist Conditions

- \( \frac{1}{\Delta k_x} > \text{FOV}_x \)
- \( \frac{1}{\Delta k_y} > \text{FOV}_y \)

Windowing

Windowing the data in Fourier space

\[ G_w(k_x, k_y) = G(k_x, k_y) W(k_x, k_y) \]

Results in convolution of the object with the inverse transform of the window

\[ g_w(x, y) = g(x, y) * w(x, y) \]
Resolution and spatial frequency

With a window of width $W_k$, the highest spatial frequency is $W_k/2$.
This corresponds to a spatial period of $2/W_k$.

$$\text{Effective Width}$$

$$w_k = \frac{1}{W_k} \int_{-\infty}^{\infty} w(x) dx$$

Example

$$w_k = \frac{1}{W_k} \int_{-\infty}^{\infty} \text{sinc}(W_k x) dx$$

$$= \frac{1}{W_k} \left[ \text{sinc}(W_k x) \right]_{-\infty}^{\infty}$$

$$= \frac{1}{W_k} \cdot \left[ k \right]_{-\infty}^{\infty}$$

$$= \frac{1}{W_k}$$

Windowing Example

$$W(k_x, k_y) = \text{rect} \left( \frac{k_x}{W_k} \right) \text{rect} \left( \frac{k_y}{W_k} \right)$$

$$w(x, y) = F^{-1} \left[ \text{rect} \left( \frac{k_x}{W_k} \right) \text{rect} \left( \frac{k_y}{W_k} \right) \right]$$

$$= W_k W_k \text{sinc}(W_k x) \text{sinc}(W_k y)$$

$$g(x,y) = g(x,y) \ast W_k W_k \text{sinc}(W_k x) \text{sinc}(W_k y)$$
Windowing Example

\[ g(x, y) = \left[ \delta(x) + \delta(x - 1) \right] \delta(y) \]

\[ g_{sw}(x, y) = \left[ \delta(x) + \delta(x - 1) \right] \delta(y) * W_s \cdot W_y \cdot \text{sinc}(W_s x) \cdot \text{sinc}(W_y y) \]

\[ = W_s W_y \left[ \delta(x) + \delta(x - 1) \right] \cdot \text{sinc}(W_s x) \cdot \text{sinc}(W_y y) \]

\[ = W_s W_y \left[ \text{sinc}(W_s x) + \text{sinc}(W_s (x - 1)) \right] \cdot \text{sinc}(W_y y) \]

Sampling and Windowing

Sampling and windowing the data in Fourier space

\[ G_{sw}(k_x, k_y) = G(k_x, k_y) \cdot \frac{1}{\Delta k_x \cdot \Delta k_y} \cdot \text{comb} \left( k_x, \frac{k_y}{W_y} \right) \cdot \text{rect} \left( k_x, \frac{k_y}{W_s} \right) \]

Results in replication and convolution in object space.

\[ g_{sw}(x, y) = W_s W_y g(x, y) * \text{comb}(\Delta k_x, \Delta k_y) * \text{sinc}(W_s x) \cdot \text{sinc}(W_y y) \]

Sampling in k_y

\[ \Delta k_y = \frac{\gamma}{2\pi} G_y \tau_y \]

\[ FOV_y = \frac{1}{\Delta k_y} \]
Sampling in $k_x$

RF Signal

- $\cos \omega_0 t$
- $\sin \omega_0 t$

Low pass Filter

ADC $\rightarrow$ I

ADC $\rightarrow$ Q

Note: In practice, there are number of ways of implementing this processing.

One I,Q sample every $\Delta t$

$M = I + jQ$

Resolution

$\delta_x = \frac{1}{W_{k_x}} = \frac{1}{2k_{x,\text{max}}} = \frac{1}{2\pi G_x \tau_x}$

$\delta_y = \frac{1}{W_{k_y}} = \frac{1}{2k_{y,\text{max}}} = \frac{1}{2\pi G_y \tau_y}$

Example

Goal:

$FOV_x = FOV_y = 25.6 \text{ cm}$

$\delta_x = \delta_y = 0.1 \text{ cm}$

Readout Gradient:

$G_x = \frac{G_y \tau_x}{2\pi}$

$\Delta t = 32 \mu s$

$G_y = \frac{\frac{1}{FOV_y} \frac{1}{\gamma} \frac{1}{G_x \tau_x}}{\frac{1}{2\pi}}$

$G_y = 2.8675 \times 10^{-5} \frac{T}{\text{cm}}$

$\tau_y = \frac{1}{G_y}$

$1 \text{ Gauss} = 1 \times 10^{-4} \text{ Tesla}$
Example

Readout Gradient:
\[
\delta_x = \frac{1}{2\pi} \frac{1}{G_x \tau_x}
\]
\[
\tau_x = \frac{1}{\delta_x} \frac{1}{2\pi} \frac{1}{G_x}
\]
\[
= \frac{1}{0.1 \text{cm}} \cdot \frac{4257 \text{ G/s}}{0.28675 \text{ G/cm}}
\]
\[
= 8.192 \text{ ms}
\]
where
\[
N_{\text{read}} = \frac{\text{FOV}}{\delta_x} = 256
\]

Example

Phase - Encode Gradient:
\[
\delta_y = \frac{1}{2\pi} \frac{1}{2G_y \tau_y}
\]
\[
G_y = \frac{1}{\delta_y} \frac{1}{2\pi} \frac{1}{2G_y}
\]
\[
= \frac{1}{0.1 \text{cm}} \cdot \frac{4257 \text{ G/s}}{0.28675 \text{ G/cm}}
\]
\[
= 0.2868 \text{ G/cm}
\]
where
\[
N_y = \frac{\text{FOV}}{\delta_y} = 256
\]

Example

In practice, an even number (typically power of 2) sample is usually taken in each direction to take advantage of the Fast Fourier Transform (FFT) for reconstruction.

Sampling
Consider the k-space trajectory shown below. ADC samples are acquired at the points shown with $\Delta t = 10\, \mu\text{sec}$. The desired FOV (both x and y) is 10 cm and the desired resolution (both x and y) is 2.5 cm. Draw the gradient waveforms required to achieve the k-space trajectory. Label the waveform with the gradient amplitudes required to achieve the desired FOV and resolution. Also, make sure to label the time axis correctly.

**Example**

\[ \text{Gibbs Artifact} \]

- 256x256 image
- 256x128 image

[Images from http://www.mritutor.org/mritutor/gibbs.htm]

**Apodization**

\[ h(k_x) = \frac{1}{2}(1 + \cos(2\pi k_x)) \]

\[ 0.5\text{sinc}(x) + 0.25\text{sinc}(x-1) + 0.25\text{sinc}(x+1) \]

[Images from http://www.mritutor.org/mritutor/gibbs.htm]
Temporal filtering in the readout direction limits the readout FOV. So there should never be aliasing in the readout direction.

Lowpass filter in the readout direction to prevent aliasing.